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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

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ON A CLASS OF QUADRATIC POLYNOMIALS IN n VARIABLES

By D. M. E. FOSTER (*London*)

[Received 10 November 1956; in revised form 7 June 1957]

1. LET $L_1,\dots, L_n$ denote n homogeneous linear forms in n variables $u_1,\dots, u_n$ with real coefficients and determinant $\Delta \neq 0$, and let c be any real number. We write

$$\wp = \wp(u_1,\dots,u_n) = \pm L_1^2 \pm \dots \pm L_{n-1}^2 + L_n + c, \tag{1}$$

$$q = q(u_1,\dots,u_n) = \pm L_1^2 \pm \dots \pm L_{n-1}^2, \tag{2}$$

where q is a singular quadratic form with rank $n-1$ and signature $s = 2r+1-n$, r being the number of positive signs in (2). If $\wp$ and $\wp'$ are any two such polynomials, we say that they are *equivalent*, writing $\wp \sim \wp'$, if they can be transformed into one another by an integral, inhomogeneous, unimodular substitution of the variables $u_1,\dots, u_n$. From his general theorem (6) on infinite convex regions Macbeath has deduced, with a slightly different normalization, that, if $|s| = n-1$, the inequality

$$|\wp| < (C_n |\Delta|)^{2/(n+1)} \tag{3}$$

is soluble in integers $u_1,\dots, u_n$ with

$$C_n^{\frac{1}{2}(n+1)} = 2^{\frac{1}{2}(n-5)}\pi^{-\frac{1}{2}(n-1)}(n+1)\Gamma\{\tfrac{1}{2}(n+1)\}. \tag{4}$$

This constant is not best-possible and indeed, for large values of n, a substantial improvement can be obtained by a modification of his argument.† The object of this paper is to prove the following two theorems.

THEOREM 1. *If the coefficients of $\wp - c$ are not all in a rational ratio, $\wp$ assumes arbitrarily small values for integers $u_1,\dots, u_n$.*

THEOREM 2. *If the coefficients of $\wp - c$ are in a rational ratio, then there are integers $u_1,\dots, u_n$ satisfying*

$$|\wp| \leqslant (\tfrac{1}{2}|\Delta|)^{2/(n+1)} \tag{5}$$

except possibly when $|s| = n-1$ and $n \geqslant 10$.

We observe that when

$$\wp = \wp_0 = \pm u_1(u_1+1) \pm \dots \pm u_{n-1}(u_{n-1}+1) + 2u_n + 1,$$

† An application of Macbeath's theorem (6), combined with Blichfeldt's classical estimate for the critical determinant of an n-dimensional sphere, leads to a constant $C_n' \sim \frac{1}{2}C_n \sim n/2\pi e$, as $n \to \infty$.

Quart. J. Math. Oxford (2), 9 (1958), 241-56.

the equality sign in (5) is necessary. For $\wp_0 \equiv 1 \pmod 2$ when $u_1,\ldots, u_n$ are integers, and so

$$|\wp_0| \geqslant 1 = (\tfrac{1}{2}|\Delta|)^{2/(n+1)}.$$

The proof of each theorem is to a large extent by induction on n, starting with the case $n = 2$. This was studied by Macbeath (**5**), who obtained, in addition, some isolation results for the minimum of a quadratic polynomial in two variables. For completeness, I prove both theorems for $n = 2$ in § 2. As in the general case these proofs depend on Lemmas 1 and 2. Lemma 1 is stated without proof and is a consequence of a general theorem (**10**) due to Weyl on the uniformity of distribution of the quadratic polynomial $\theta_1 x^2+\theta_2 x+\theta_3 \pmod 1$. Lemma 2, which gives an inequality for a quadratic in a single integral variable, is a corollary of Lemma 5 given by Davenport (**3**), but for convenience I give a proof in § 2. The proof of Theorem 2, for general n [see § 3], is based upon the fact that a suitable integral unimodular substitution of the variables $u_1,\ldots, u_n$ reduces $q(u_1,\ldots, u_n)$ to a *non-singular* quadratic form in $n-1$ variables of determinant D say, with the property that its first coefficient (say a) satisfies

$$0 < |a| \leqslant 2|D|^{1/(n-1)}.$$

In the final section, § 4, I prove Theorem 1, the proof depending mainly on Lemmas 1 and 5, and at one point on Theorem 2.

I am very grateful to Dr. J. H. H. Chalk for all his help and encouragement during my work on this problem. I also wish to thank the referee for some helpful comments, including the simplification of Lemma 5.

2. LEMMA 1. *If at least one of θ_1, θ_2 is irrational, and ϵ is any assigned positive number, the inequality*

$$|\theta_1 x^2+\theta_2 x+y+\theta_3| < \epsilon$$

has an infinity of solutions in integers x, y.

LEMMA 2. *If $n \geqslant 2$ and a, α, t are any real numbers satisfying*

$$0 < a \leqslant 2, \tag{6}$$

$$0 \leqslant t^2 \leqslant \tfrac{1}{4}a^2-a+2a^{1-1/n}, \tag{7}$$

then there is an integer u for which

$$|a(u+\alpha)^2-a^{-1}t^2| \leqslant 1. \tag{8}$$

Proof. Put $f(u) = a(u+\alpha)^2-a^{-1}t^2$

for convenience. Since

$$2a^{-1}(t^2+a)^{\frac{1}{2}} \geqslant 2a^{-\frac{1}{2}} \geqslant \sqrt{2} > 1,$$

by (6) and (7), we can choose an integer u which satisfies

$$|u+\alpha| < a^{-1}(t^2+a)^{\frac{1}{2}},$$

and then $f(u)$ satisfies

$$-a^{-1}t^2 \leqslant f(u) < 1.$$

Hence, if $t^2 < a$, (8) is valid with strict inequality. Suppose now that $t^2 \geqslant a$, and let u denote the integer for which

$$a^{-1}(t^2+a)^{\frac{1}{2}}-1 \leqslant u+\alpha < a^{-1}(t^2+a)^{\frac{1}{2}}.$$

Clearly, if it also satisfies

$$u+\alpha > a^{-1}(t^2-a)^{\frac{1}{2}},$$

(8) is again satisfied with strict inequality. Hence we may suppose that

$$a^{-1}(t^2+a)^{\frac{1}{2}}-1 \leqslant u+\alpha \leqslant a^{-1}(t^2-a)^{\frac{1}{2}},$$

where

$$a^{-1}(t^2+a)^{\frac{1}{2}}-1 \leqslant a^{-1}(t^2-a)^{\frac{1}{2}}. \tag{9}$$

The condition (9) combined with (7) gives us a lower bound for a: for on rearranging (9) and squaring we have

$$2(t^4-a^2)^{\frac{1}{2}} \geqslant 2t^2-a^2,$$

where

$$2t^2-a^2 \geqslant 2a-a^2 = a(2-a) \geqslant 0.$$

Squaring once again, we then have

$$t^2 \geqslant 1+\tfrac{1}{4}a^2$$

and so, on using (7),

$$2a^{1-1/n}-a \geqslant 1,$$

which gives

$$a \geqslant 1 \tag{10}$$

since $n \geqslant 2$. Consider now $f(u-1)$. From our present choice of u, we have

$$a^{-1}(t^2+a)^{\frac{1}{2}}-2 \leqslant u-1+\alpha \leqslant a^{-1}(t^2-a)^{\frac{1}{2}}-1,$$

where

$$a^{-1}(t^2-a)^{\frac{1}{2}}-1 \leqslant \left(\frac{1}{4}-\frac{2}{a}+\frac{2}{a^{1+1/n}}\right)^{\frac{1}{2}}-1 \leqslant \tfrac{1}{2}-1 < 0,$$

by (10). Hence

$$a\{a^{-1}(t^2-a)^{\frac{1}{2}}-1\}^2-a^{-1}t^2 \leqslant f(u-1) \leqslant a\{a^{-1}(t^2+a)^{\frac{1}{2}}-2\}^2-a^{-1}t^2,$$

where

$$a\{a^{-1}(t^2+a)^{\frac{1}{2}}-2\}^2-a^{-1}t^2 = 1+4\{a-(t^2+a)^{\frac{1}{2}}\} \leqslant 1,$$

since

$$t^2 \geqslant a \geqslant a^2-a$$

follows from $a \leqslant 2$, and where

$$a\{a^{-1}(t^2-a)^{\frac{1}{2}}-1\}^2-a^{-1}t^2 = a-1-2(t^2-a)^{\frac{1}{2}} \geqslant -1$$

since

$$a+\tfrac{1}{4}a^2 \geqslant \tfrac{1}{4}a^2+2a^{1-1/n}-a \geqslant t^2$$

follows from (7) and (10).

Note that inequality in (8) occurs only when

$$\textit{either} \quad t^2 = \tfrac{5}{4}, \quad a = 1, \quad \alpha \equiv \tfrac{1}{2} \pmod 1,$$

$$\textit{or} \quad t^2 = 2, \quad a = 2, \quad \alpha \equiv 0 \pmod 1.$$

Proof of Theorems 1 *and* 2 ($n = 2$). By considering a suitable multiple of $\wp$ in place of $\wp$ if necessary, we may ssume that

$$|\Delta| = 2. \tag{11}$$

Then it is sufficient to show that $\wp(u_1, u_2)$ assumes arbitrarily small values for integers u_1, u_2 unless the coefficients of $\wp - c$ are in a rational ratio; in which case, we show that $|\wp| \leqslant 1$ is soluble in integers u_1, u_2. Let $\epsilon > 0$. Then, by Minkowski's theorem on linear forms, there are coprime integers u_1^*, u_2^* satisfying

$$|L_1| \leqslant \epsilon^{\frac{1}{2}}, \qquad |L_2| \leqslant 2\epsilon^{-\frac{1}{2}}. \tag{12}$$

After applying a suitable integral unimodular substitution to the variables u_1, u_2 we can ensure that $(u_1^*, u_2^*) = (1, 0)$, and then $\pm\wp$ can be expressed as

$$\pm\wp = a(u_1 + b_2 u_2 + b_3)^2 + c_1 u_1 + c_2 u_2 + c_3 \tag{13}$$

for suitable real a, b_2, b_3, c_1, c_2, c_3, where

$$\textit{either} \quad \text{(i)} \quad 0 < a \leqslant \epsilon, \quad c_1 = 0,$$

$$\textit{or} \quad \text{(ii)} \quad a > 0, \quad b_2 = b_3 = 0.$$

Note that (i) is an obvious consequence of (12) provided that $L_1 \neq 0$, while (ii) arises naturally when $L_1 = 0$.

Case (i). By (11), we have $a^{\frac{1}{2}}|c_2| = 2$. Choosing u_2, u_1 successively, and writing $c_2 u_2 + c_3 = -\lambda$, we can ensure that

$$0 \leqslant \lambda < |c_2| = 2a^{-\frac{1}{2}}, \tag{14}$$

$$|u_1 + b_2 u_2 + b_3 - \lambda^{\frac{1}{2}} a^{-\frac{1}{2}}| \leqslant \tfrac{1}{2}.$$

Then

$$\begin{aligned} |\wp| &\leqslant (a\lambda)^{\frac{1}{2}} + \tfrac{1}{4}a \\ &< (2a^{\frac{1}{2}})^{\frac{1}{2}} + \tfrac{1}{4}a, \end{aligned}$$

by (14), and, since $a \leqslant \epsilon$, where ϵ is arbitrary, it follows that $\wp$ assumes arbitrarily small values for integral u_1, u_2.

Case (ii). We may apply Lemma 1 directly to

$$\pm\wp = au_1^2 + c_1 u_1 + c_2 u_2 + c_3,$$

where $a^{\frac{1}{2}}|c_2| = 2$, by taking

$$\theta_1 = ac_2^{-1}, \qquad \theta_2 = c_1 c_2^{-1}.$$

Thus, unless the coefficients of $\wp - c_3$ are in a rational ratio, $\wp$ assumes

arbitrarily small values. Otherwise, on writing $\pm\wp$ in the alternative form

$$\pm\wp = a(u_1+\alpha)^2+c_2u_2+\beta$$

with appropriate α, β we can establish the required inequality $|\wp| \leqslant 1$. For, if $|c_2| < 2$, we may select any value of u_1 and then choose u_2 to satisfy $|\wp| \leqslant \frac{1}{2}|c_2| < 1$. If $|c_2| \geqslant 2$, we apply Lemma 2; note that since $a^{\frac{1}{2}}|c_2| = 2$, we have

$$a \leqslant 1. \tag{15}$$

Writing $-\lambda = c_2u_2+\beta$ for convenience, we choose u_2 first to satisfy

$$\tfrac{1}{4}a-1 < \lambda \leqslant \tfrac{1}{4}a-1+2a^{-\frac{1}{2}}. \tag{16}$$

Then, if $\lambda < 0$, the integer u_1 may be selected so that

$$|u_1+\alpha| \leqslant \tfrac{1}{2}$$

and then

$$|\wp| < \tfrac{1}{4}a+(1-\tfrac{1}{4}a) = 1.$$

But, if $\lambda \geqslant 0$, we may introduce $t^2 = a\lambda$ and then the choice of u_1 in Lemma 2 ensures that $|\wp| \leqslant 1$, by (15) and (16); and in fact we again have strict inequality unless

$$\wp \sim C(u_1^2+u_1+2u_2-1).$$

3. For the proof of Theorem 2 when $n > 2$ we require two further lemmas.

Definition. Two quadratic forms q, q' in n variables are *equivalent* if they can be transformed into each other by a homogeneous integral unimodular substitution on the variables.

LEMMA 3. *If $q = q(u_1,\ldots,u_n)$ is a singular quadratic form in the n variables $u_1,\ldots,u_n$ with rational coefficients and rank $n-1$, it is equivalent to a non-singular quadratic form in $n-1$ variables with rational coefficients.*

Proof. By Lagrange's reduction of a quadratic form we can express q in the form

$$q = a_1L_1^2+\ldots+a_{n-1}L_{n-1}^2,$$

where $a_1,\ldots,a_{n-1}$ are rational non-zero numbers and $L_1,\ldots,L_{n-1}$ are $n-1$ independent linear forms in $u_1,\ldots,u_n$ with *rational* coefficients. If we write

$$L_r = \sum_{s=1}^{n} \alpha_{rs}u_s \quad (r = 1,\ldots,n-1),$$

where the α_{rs} are rational, we may suppose, by permuting the variables, that

$$||\alpha_{rs}||_{r,s=1,\ldots,n-1} \neq 0.$$

Then there exist rational numbers $u_1,\ldots,u_{n-1}$ satisfying the equations

$$L_1 = \ldots = L_{n-1} = 0 \tag{17}$$

for any prescribed rational value of u_n. Since the equations (17) are

linear, it follows that they have a solution in coprime integers $u_1^*,\ldots, u_n^*$, and, if we now apply a suitable unimodular substitution to the variables $u_1,\ldots, u_n$, we may suppose that $(u_1^*,\ldots, u_{n-1}^*, u_n^*) = (0,\ldots, 0, 1)$. Hence $L_1,\ldots, L_{n-1}$ reduce to linear forms in $n-1$ variables with rational coefficients, and q is of the required form.

LEMMA 4. *If $q = q(u_1,\ldots, u_n)$ is an indefinite quadratic form in n variables $u_1,\ldots, u_n$ with rational coefficients and determinant D, then there are integers $u_1,\ldots, u_n$ satisfying*

$$0 < q \leqslant 2|D|^{1/n}.$$

Remark. I give the proof in two parts. The first is elementary and gives the required inequality for forms with small signature. This partial result interested Dr. G. L. Watson and he has recently found a proof of a rather better inequality valid for any signature. A brief outline of his argument is given in the second part of the proof. [See G. L. Watson, *Quart. J. of Math.* (Oxford) (2) 9 (1958) 99–108 for details.] This proof is not elementary and it would be of interest to find a direct proof† of Lemma 4.

Proof. Let the signature be s.

(i) Suppose $|s| \leqslant 8$. The lemma has been established for $n = 2$, 3, and 4 by Segre (**9**), Davenport (**4**), and Oppenheim (**7**), without the restriction that the forms be rational. Thus it is sufficient to consider values of $n \geqslant 5$, and the proof is inductive, depending on the cases $n = 3$ and 4. By considering a suitable multiple of q in place of q, if necessary, we may suppose that $|D| = 1$, and it suffices to show that the inequalities

$$0 < q \leqslant 2 \tag{18}$$

are soluble in integers $u_1,\ldots, u_n$. By a classical theorem of Meyer's, q represents zero for relatively prime integers $u_1^*,\ldots, u_n^*$ and after applying a suitable integral unimodular substitution to the variables $u_1,\ldots, u_n$ we can take $(u_1^*, u_2^*,\ldots, u_n^*) = (1, 0,\ldots, 0)$. Then

$$q \sim au_1(\alpha_2 u_2+\ldots+\alpha_n u_n)+q_1(u_2,\ldots, u_n)$$

for appropriate rational $a \neq 0$, relatively prime integers $\alpha_2,\ldots, \alpha_n$, and q_1. By a further integral unimodular substitution of the type

$$u_2' = \alpha_2 u_2+\ldots+\alpha_n u_n,$$

$$u_r' = \alpha_2^{(r)}u_2+\ldots+\alpha_n^{(r)}u_n \quad (r = 3,\ldots, n),$$

we see that

$$q \sim au_1 u_2+q_2(u_2,\ldots, u_n),$$

or

$$q \sim au_2\{u_1+b_2 u_2+b_3(\beta_3 u_3+\ldots+\beta_n u_n)\}+q_3(u_3,\ldots, u_n),$$

† It seems difficult to modify our argument for $|s| \geqslant 9$.

for appropriate b_2, b_3, relatively prime integers $\beta_3,\ldots,\beta_n$ and q_3. With a third integral unimodular substitution, on the variables $u_3,\ldots,u_n$ we have

$$q \sim au_2(u_1+b_2u_2+b_3u_3)+q_4'(u_3,\ldots,u_n), \tag{19}$$

where $q_4 = q_4(u_3,\ldots,u_n)$ is a quadratic form in the $n-2$ variables $u_3,\ldots,u_n$ with rational coefficients and determinant $\bar{D}$, say. Comparing determinants we see that

$$|\bar{D}| = 4a^{-2}. \tag{20}$$

By changing the sign of u_2, if necessary, we can ensure that

$$a > 0.$$

If $a \leqslant 2$, let $\qquad (u_2, u_3,\ldots,u_n) = (1, 0,\ldots, 0)$

and choose u_1 to be an integer satisfying

$$0 < a(u_1+b_2) \leqslant a,$$

which gives (18). If $a > 2$, then

$$|\bar{D}| < 1, \tag{21}$$

by (20). Since the signature of q is an algebraic invariant, it follows, by (19), that q_4 has the same signature as q. Since the lemma holds for $n = 2, 3, 4$, we suppose that it is true for indefinite quadratic forms in at most $n-1 \geqslant 4$ variables, and then prove it true for indefinite quadratic forms in n variables. Thus the result will follow by induction on n. Since $|s| \leqslant 8$, $q_4(u_3,\ldots,u_n)$ is a quadratic form in $n-2$ variables which is indefinite for $n \geqslant 11$ but may be either indefinite or definite when $5 \leqslant n \leqslant 10$. If $q_4(u_3,\ldots,u_n)$ is indefinite, there are integers $u_3,\ldots,u_n$ satisfying

$$0 < q_4(u_3,\ldots,u_n) \leqslant 2|\bar{D}|^{1/(n-2)} \leqslant 2, \tag{22}$$

by the inductive hypothesis, which gives (18) with $u_1 = u_2 = 0$. If, however, $q_4(u_3,\ldots,u_n)$ is definite, in which case $5 \leqslant n \leqslant 10$, it is well known (1) by classical estimates that there are integers $u_3,\ldots,u_n$ satisfying (22) with q_4 replaced by $|q_4|$. Then (18) is again true with $u_1 = u_2 = 0$ unless q_4 is negative definite. In this case there are relatively prime integers

$$(u_1, u_2, u_3,\ldots,u_n) = (0, 0, u_3^*,\ldots,u_n^*)$$

for which $q = -b$ say, where

$$0 < b \leqslant 2. \tag{23}$$

If we now apply a suitable unimodular substitution to $u_1,\ldots,u_n$, we may write

$$q = -b(u_1+\alpha_{12}u_2+\ldots+\alpha_{1n}u_n)^2+q'(u_2,\ldots,u_n),$$

where $q' = q'(u_2,..., u_n)$ is an indefinite quadratic form in $u_2,..., u_n$ of determinant

$$D' = \pm b^{-1}. \tag{24}$$

Then there are integers $u_2^*,..., u_n^*$ for which $q' = \lambda$, where

$$0 < \lambda \leqslant 2b^{-1/(n-1)}, \tag{25}$$

by the inductive hypothesis. Hence

$$q(u_1, u_2^* u_2,..., u_n^* u_2) = -b(u_1+\alpha u_2)^2+\lambda u_2^2,$$

for some α. By the case $n = 2$ of the lemma we can satisfy

$$0 < -b(u_1+\alpha u_2)^2+\lambda u_2^2 \leqslant 2(b\lambda)^{\frac{1}{2}}$$

in integers u_1, u_2; this is sufficient for our purpose unless

$$\lambda b > 1. \tag{26}$$

Thus we suppose (26) true. If $\lambda \leqslant 2$, we choose an integer u_1 to satisfy

$$|u_1+\alpha| \leqslant \tfrac{1}{2},$$

and, since $\lambda-2 \leqslant 0, \quad \frac{1}{4} \leqslant b^{-2} \leqslant b^{-1}\lambda,$

by (23) and (26), we have

$$b^{-1}(\lambda-2) \leqslant (u_1+\alpha)^2 < b^{-1}\lambda, \tag{27}$$

which gives (18) with $u_2 = 1$. If $\lambda > 2$, (25) implies that

$$0 < b < 1.$$

We therefore have the inequality

$$(b+2)^2 > 8b^{1-1/(n-1)}, \quad \text{if } n \geqslant 5,$$

and so, by (25), $\lambda < (b+2)^2/4b.$

Thus $$\lambda^{\frac{1}{2}}+(\lambda-2)^{\frac{1}{2}} < \frac{b+2}{2b^{\frac{1}{2}}}+\frac{2-b}{2b^{\frac{1}{2}}} = 2b^{-\frac{1}{2}},$$

and so $$\lambda^{\frac{1}{2}}-(\lambda-2)^{\frac{1}{2}} = 2/\{\lambda^{\frac{1}{2}}+(\lambda-2)^{\frac{1}{2}}\} > b^{\frac{1}{2}}.$$

Hence there is an integer u_1 satisfying (27). The lemma is now established when $|s| \leqslant 8$.

(ii) *General case.* Let q be expressed in the form

$$q = \xi_1^2+...+\xi_r^2-\xi_{r+1}^2-...-\xi_{r+t}^2 \quad (n = r+t, s = r-t),$$

where $\xi_1,..., \xi_n$ are real linear forms in the variables $u_1,..., u_n$. Then we say that q is of *type* (r, t). Let (r', t') be any positive integers satisfying

$$r' \equiv r \pmod 4, \quad t' \equiv t \pmod 4, \quad r'+t' = n.$$

Then the following principle due to Watson is sufficient for our purpose.

For $n \geqslant 4$, there exists a form q' in the n variables $u_1,..., u_n$ with the same determinant as q and of type (r', t') with the property that it represents, for integral values of the variables, the same set of values as q.

Thus, in particular, there is such a form q' with signature s' satisfying $|s'| \leqslant 4$, and clearly it must be indefinite.

Proof of Theorem 2 *when* $n > 2$. We suppose that the theorem is valid for polynomials in $n-1$ variables, and then prove it valid for n variables. Since the theorem has been established for two variables, the result follows by induction. There is again no loss of generality in taking $|\Delta| = 2$, and it now suffices to show that there are integers $u_1,..., u_n$ for which

$$|\wp| \leqslant 1 \tag{28}$$

provided that $n \leqslant 9$ when $|s| = n-1$. By Lemma 3, we may suppose that $L_1,..., L_{n-1}$ are linear forms in the $n-1$ variables $u_1,..., u_{n-1}$ with determinant $\Delta' \neq 0$. If $|s| = n-1 \leqslant 8$, it is well known (1) that we can apply a further unimodular substitution to the variables $u_1,..., u_{n-1}$ to ensure that the coefficient of u_1^2, say a, satisfies

$$0 < \kappa a \leqslant 2|\Delta'|^{2/(n-1)}, \tag{29}$$

where

$$\kappa = \begin{cases} 1 & \text{if } s = n-1, \\ -1 & \text{if } s = -(n-1). \end{cases} \tag{30}$$

Otherwise, if $|s| \neq n-1$, we can ensure that a satisfies (29) with

$$\kappa = \begin{cases} 1 & \text{if } -(n-3) < s \leqslant n-3, \\ -1 & \text{if } s = -(n-3), \quad \text{by Lemma 4.} \end{cases} \tag{31}$$

Thus in either case we may suppose, by considering $-\wp$ in place of $\wp$ if necessary, that

$$0 < a \leqslant 2|\Delta'|^{2/(n-1)}. \tag{32}$$

We write

$$\wp = a\bar{L}_1^2 \pm \bar{L}_2^2 \pm ... \pm \bar{L}_{n-1}^2 + \bar{L}_n + c,$$

where

$$\bar{L}_r = \sum_{s=1}^{n-1} \alpha_{rs} u_s \quad (r = 1,..., n-1), \qquad \bar{L}_n = \sum_{s=1}^{n} \alpha_{ns} u_s,$$

$$\alpha_{11} = 1, \qquad \alpha_{r1} = 0 \quad (r = 2,..., n-1).$$

To evaluate the absolute value of the determinant $\bar{\Delta}$, say, of the linear forms $a^{\frac{1}{2}}\bar{L}_1$, $\bar{L}_2,..., \bar{L}_n$ we consider the quadratic form

$$\pm L_1^2 \pm ... \pm L_{n-1}^2 + L_n u_{n+1} + c u_{n+1}^2$$

in the original n variables $u_1,..., u_n$ and a new variable u_{n+1}. Its determinant ± 1 is clearly left invariant under unimodular substitutions applied to $u_1,..., u_n$. Thus the quadratic form

$$a\bar{L}_1^2 \pm \bar{L}_2^2 \pm ... \pm \bar{L}_{n-1}^2 + \bar{L}_n u_{n+1} + c u_{n+1}^2$$

also has determinant $\pm\frac{1}{4}\bar{\Delta}^2 = \pm 1$, and so $|\bar{\Delta}| = 2$. Hence, comparing determinants,

$$|\Delta' \alpha_{nn}| = 2. \tag{33}$$

It is convenient to express $\wp$ in the form

$$\wp = a(\bar{L}_1+b)^2 \pm \bar{L}_2^2 \pm \ldots \pm \bar{L}_{n-1}^2 + L_n' + c',$$

where the absolute value of the determinant of $\bar{L}_2, \ldots, \bar{L}_{n-1}, L_n'$ in the $n-1$ variables $u_2, \ldots, u_n$ is

$$a^{-\frac{1}{2}}|\Delta'\alpha_{nn}| = 2a^{-\frac{1}{2}},$$

by (33). The theorem is trivial if $|\alpha_{nn}| \leqslant 2$ since for any values of $u_1, \ldots, u_{n-1}$ we can always choose an integer u_n so that

$$|\wp| \leqslant \tfrac{1}{2}|\alpha_{nn}| \leqslant 1.$$

We therefore suppose that $|\alpha_{nn}| > 2$, which implies that $|\Delta'| < 1$, by (33), and hence

$$0 < a < 2. \tag{34}$$

Consider the polynomial

$$\bar{\wp} = \bar{\wp}(u_2, \ldots, u_n) = \pm \bar{L}_2^2 \pm \ldots \pm \bar{L}_{n-1}^2 + L_n' + c'.$$

If $|s| = n-1 \leqslant 8$, the quadratic section of $\bar{\wp}$ is a definite quadratic form in $u_2, \ldots, u_n$, while, if $|s| \neq n-1$, our choice of a in (29) and (31) ensures that the quadratic section of $\bar{\wp}$ is an *indefinite* quadratic form in $u_2, \ldots, u_{n-1}$. In either case, therefore, we can apply the inductive hypothesis to $\bar{\wp}$ to establish the existence of integers $u_2, \ldots, u_n$ for which

$$\bar{\wp} = -\lambda,$$

where

$$\tfrac{1}{4}a - 1 < \lambda \leqslant \tfrac{1}{4}a - 1 + 2(\tfrac{1}{2}.2a^{-\frac{1}{2}})^{2/n}.$$

With these values of $u_2, \ldots, u_n$ we have

$$\wp = a(u_1+\alpha)^2 - \lambda, \tag{35}$$

for an appropriate value of α. If

$$\tfrac{1}{4}a - 1 < \lambda < 0,$$

we can always choose an integer u_1 so that

$$|u_1+\alpha| \leqslant \tfrac{1}{2},$$

and then

$$|\wp| < \tfrac{1}{4}a + (1 - \tfrac{1}{4}a) = 1.$$

Thus we can suppose that

$$0 \leqslant \lambda \leqslant \tfrac{1}{4}a - 1 + 2a^{-1/n},$$

or, if $t^2 = a\lambda$, that

$$0 \leqslant t^2 \leqslant \tfrac{1}{4}a^2 - a + 2a^{1-1/n}. \tag{36}$$

Hence, by Lemma 2, there is an integer u_1 satisfying

$$|a(u_1+\alpha)^2 - a^{-1}t^2| \leqslant 1,$$

by (34), (35), (36). This completes the proof of the theorem.

Remark. If $|s| = n-1 \geqslant 9$, the argument above gives a constant

$$\left\{\frac{1}{2}\left(\frac{\gamma_{n-1}}{2}\right)^{\frac{1}{2}(n-1)}\right\}^{2/(n+1)}$$

in place of $2^{-2/(n+1)}$ on the right-hand side of (5). This is asymptotically equivalent to Macbeath's constant $C_n^{2/(n+1)}$ in (3), for large values of n, when we introduce Blichfeldt's estimate for γ_{n-1}.

4. Before proving the general case of Theorem 1 we require a further lemma.

LEMMA 5. *For $n \geqslant 3$ suppose that the equations $L_1 = \ldots = L_{n-1} = 0$ have no solution in integers $u_1,\ldots, u_n$ other than $0,\ldots, 0$. Then the singular quadratic form*

$$\pm L_1^2 \pm \ldots \pm L_{n-1}^2$$

in the variables $u_1,\ldots, u_n$ assumes an arbitrarily small non-zero value for integers $u_1,\ldots, u_n$.

Proof. If we write

$$L_r = \sum_{s=1}^{n} \alpha_{rs} u_s \quad (r = 1,\ldots, n-1), \tag{37}$$

the matrix $(\alpha_{rs})_{n-1,n}$ has rank $n-1$, by hypothesis, and we may suppose, without loss of generality, that

$$\Delta_1 = \|\alpha_{rs}\|_{r,s=1,\ldots,n-1} \neq 0.$$

For any $\epsilon > 0$, there are an infinity of integers $u_1,\ldots, u_n$ satisfying

$$|L_i| \leqslant (\epsilon/n-1)^{\frac{1}{2}} \quad (i = 1,\ldots, n-1), \qquad |u_n| \leqslant (\epsilon/n-1)^{-(n-1)/2}|\Delta_1|, \tag{38}$$

by Minkowski's theorem on linear forms and our hypothesis concerning $L_1,\ldots, L_{n-1}$. Hence

$$|\pm L_1^2 \pm \ldots \pm L_{n-1}^2| \leqslant \epsilon$$

for integers $u_1,\ldots, u_n$ and, unless

$$\pm L_1^2 \pm \ldots \pm L_{n-1}^2 = 0,$$

when $$\pm L_1^2 \pm \ldots \pm L_{n-1}^2 \neq \pm(L_1^2 + \ldots + L_{n-1}^2),$$

there is nothing further to prove. Suppose that there are integers

$$(u_1,\ldots, u_n) \neq (0,\ldots, 0)$$

for which $$\pm L_1^2 \pm \ldots \pm L_{n-1}^2 = 0.$$

Consider first the cases $n = 3$ and 4.

If $n = 3$, it is well known (2) that there are integers u_1, u_2, u_3 satisfying

$$0 < (L_1 - L_2)(L_1 + L_2) < \epsilon.$$

If $n = 4$, each of the integer-vectors $u_1,\ldots, u_4$ satisfying (38) gives rise to a point $(x_1, x_2, x_3) = (L_1, L_2, L_3)$ in 2-dimensional projective space S_2. Two at least of these points, say $P(\xi_1, \xi_2, \xi_3)$ and $Q(\eta_1, \eta_2, \eta_3)$, must be distinct, and, unless they both lie on the conic

$$x_1^2 \pm (x_2^2 - x_3^2) = 0,$$

there is nothing further to prove. If, however, they both lie on this conic, then $P+Q$ does not, for we cannot have three collinear points on a conic, and hence

$$0 < |(\xi_1+\eta_1)^2 \pm \{(\xi_2+\eta_2)^2 - (\xi_3+\eta_3)^2\}| \leqslant 4\epsilon.$$

In other words, there are integers $u_1,\ldots, u_4$ satisfying

$$0 < |L_1^2 \pm (L_2^2 - L_3^2)| \leqslant 4\epsilon,$$

which is again the required result.

We now suppose that $n \geqslant 5$ and, by interchanging the L_i $(i = 1,\ldots, n-1)$ if necessary, that

$$\pm L_1^2 \pm \ldots \pm L_{n-1}^2 = \pm(L_1^2 \pm \ldots - L_{n-1}^2) = 0$$

for relatively prime integers $(u_1^*,\ldots, u_n^*)$. By applying a suitable integral unimodular substitution to $u_1,\ldots, u_n$ we may assume further that

$$(u_1^*, u_2^*,\ldots, u_n^*) = (1, 0,\ldots, 0).$$

If we again write

$$L_r = \sum_{s=1}^{n} \alpha_{rs} u_s \quad (r = 1,\ldots, n-1),$$

where the α_{rs} are not necessarily the same as before, the rank of the matrix $(\alpha_{rs})_{n-1,n}$ is $n-1$, by hypothesis. Let $\mathbf{a}_i$ $(i = 1,\ldots, n)$ denote the ith column vector in (α_{rs}); and let A_i $(i = 1,\ldots, n)$ denote the $n-1 \times n-1$ matrices formed from (α_{rs}) by omitting the ith column. Since the rank of (α_{rs}) is $n-1$, at least one of $|A_i|$ $(i = 1,\ldots, n)$ does not vanish. Consider the effect on (α_{rs}) of a substitution of the type

$$u_r = u_r' + \delta_{ri} u_j' \quad (r = 1,\ldots, n), \tag{39}$$

where $1 \leqslant i, j \leqslant n \quad (i \neq j)$, $\delta_{ri} = \begin{cases} 1 & \text{if } r = i, \\ 0 & \text{otherwise,} \end{cases}$ (40)

applied to the forms $L_1,\ldots, L_{n-1}$. For convenience of notation, denote the new matrix of coefficients by (α_{rs}'), and use $\mathbf{a}_i'$, A_i' with their natural meanings. Then clearly $\mathbf{a}_k' = \mathbf{a}_k$ for $k \neq i$, and $\mathbf{a}_j' = \mathbf{a}_j + \mathbf{a}_i$; and so

$$|A_k'| = |A_k| \quad (k \neq i), \qquad |A_i'| = |A_i| \pm |A_j|.$$

If, for any $i \geqslant 2$, $|A_i| = 0$, we can find j so that $|A_j| \neq 0$, and then the substitution (39) makes $|A_i'| \neq 0$.

We are allowed to perform any substitution of type (39) with $i \geqslant 2$, and so it follows that the set of forms given by (37) may be replaced by an equivalent set for which

$$\alpha_{11}^2 \pm \ldots - \alpha_{n-1,1}^2 = 0, \qquad |A_k| \neq 0 \quad (k = 2,\ldots,n).$$

If we put $u_n = 0$, $L_1^2 \pm \ldots - L_{n-1}^2$ is a quadratic form in $n-1 \geqslant 4$ variables $u_1,\ldots, u_{n-1}$ with determinant $|A_n| \neq 0$. Moreover it represents zero for integers $(u_1, u_2,\ldots, u_{n-1}) = (1, 0,\ldots, 0) \neq (0,\ldots, 0)$ and so it is *indefinite*. Hence, by theorems given by Oppenheim (8), it represents arbitrarily small non-zero values unless it is a multiple of a form with rational coefficients. In the latter case, all the terms in the original form $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ which do not involve u_n are in a rational ratio. If we repeat the argument with $u_{n-1} = 0$ and with $u_{n-2} = 0$, since there is no term in $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ involving three variables, we deduce that either $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ assumes an arbitrarily small non-zero value for integers $u_1,\ldots, u_n$ or it is equivalent to a multiple of a rational form in n variables. In the latter case, by Lemma 3, $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ is equivalent to a multiple of a non-singular rational quadratic form in $n-1$ variables, and we now show that this is impossible by our hypothesis concerning $L_1,\ldots, L_{n-1}$. For, after applying a suitable unimodular substitution to the variables $u_1,\ldots, u_n$ we may assume that $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ is a non-singular quadratic form in the $n-1$ variables $u_1,\ldots, u_{n-1}$. We may write

$$L_r = \sum_{s=1}^{n} \alpha_{rs} u_s \quad (r = 1,\ldots, n),$$

where $\Delta = \|\alpha_{rs}\|_{r,s=1,\ldots,n} \neq 0$. Then, by differentiating $\pm(L_1^2 \pm \ldots - L_{n-1}^2)$ partially with respect to u_n, we have

$$\alpha_{1n} L_1 \pm \alpha_{2n} L_2 \pm \ldots - \alpha_{n-1,n} L_{n-1} = 0$$

since the quadratic form, when expressed in terms of the u_i $(i = 1,\ldots, n)$, is independent of u_n. Hence

either (i) $L_1,\ldots, L_{n-1}$ are linearly dependent

or (ii) $\alpha_{1n} = \alpha_{2n} = \ldots = \alpha_{n-1,n} = 0.$

Clearly (i) is impossible since $\Delta \neq 0$. In (ii), none of the forms $L_1,\ldots, L_{n-1}$ involve u_n, and so the equations $L_1 = \ldots = L_{n-1} = 0$ are soluble with

$$(u_1,\ldots, u_{n-1}, u_n) = (0,\ldots, 0, 1),$$

which is contrary to the hypothesis of the lemma. This completes the proof.

Proof of Theorem 1 *when* $n > 2$. As in the proof of Theorem 2 we suppose that the theorem is true for $n-1$ variables and then prove it is true for n variables, the case $n = 2$ having been established independently. The proof divides into two parts according as the $n-1$ equations

$$L_1 = \ldots = L_{n-1} = 0 \tag{41}$$

have, or do not have, a non-trivial solution in integers $u_1, \ldots, u_n$.

If the equations (41) have a non-trivial solution in integers $u_1^*, \ldots, u_n^*$, which we may take to be relatively prime, we may suppose, by applying a suitable unimodular substitution to the variables $u_1, \ldots, u_n$ that

$$(u_1^*, \ldots, u_{n-1}^*, u_n^*) = (0, \ldots, 0, 1).$$

Then $L_1, \ldots, L_{n-1}$ are linear forms in the $n-1$ variables $u_1, \ldots, u_{n-1}$ only. We may write

$$\wp = q(u_1, \ldots, u_{n-1}) + L_n + c, \tag{42}$$

where

$$q(u_1, \ldots, u_{n-1}) = \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} \alpha_{rs} u_r u_s \quad (\alpha_{rs} = \alpha_{sr}),$$

$$L_n = \sum_{s=1}^{n} \alpha_{ns} u_s,$$

and, if $D = \|\alpha_{rs}\|_{r,s=1,\ldots,n-1}$, comparing determinants we see that

$$\pm\Delta = |D| \alpha_{nn}.$$

Since $\Delta \neq 0$, by hypothesis, it follows that $\alpha_{nn} \neq 0$. If we take all the variables to be zero except u_r, u_s, u_n, where $1 \leqslant r, s \leqslant n-1$, we have

$$\wp = \alpha_{rr} u_r^2 + 2\alpha_{rs} u_r u_s + \alpha_{ss} u_s^2 + \alpha_{nr} u_r + \alpha_{ns} u_s + \alpha_{nn} u_n + c.$$

Now put $u_s = 0$. Then, by Lemma 1 with

$$\theta_1 = \alpha_{nn}^{-1} \alpha_{rr}, \qquad \theta_2 = \alpha_{nn}^{-1} \alpha_{nr},$$

it follows that $\wp$ assumes an arbitrarily small value for integers u_r, u_n unless the ratios

$$\alpha_{nn}^{-1} \alpha_{rr}, \qquad \alpha_{nn}^{-1} \alpha_{nr} \qquad (r = 1, \ldots, n-1)$$

are all rational. We therefore suppose that this is the case and now take $u_r = u_s$. By a further application of Lemma 1 with

$$\theta_1 = \alpha_{nn}^{-1}(\alpha_{rr} + 2\alpha_{rs} + \alpha_{ss}), \qquad \theta_2 = \alpha_{nn}^{-1}(\alpha_{nr} + \alpha_{ns})$$

we deduce that $\wp$ again assumes an arbitrarily small value for integers u_r, u_n unless the ratios

$$\alpha_{nn}^{-1} \alpha_{rs}, \quad \alpha_{nn}^{-1} \alpha_{nr} \qquad (r, s = 1, \ldots, n-1)$$

are all rational. However, this is impossible since the coefficients of $\wp - c$ are not all in a rational ratio, and we therefore have the required result.

If the equations (41) have no solution in integers $(u_1,..., u_n)$ other than $(0,..., 0)$, then, by Lemma 5, $\pm L_1^2 \pm ... \pm L_{n-1}^2$ assumes an arbitrarily small non-zero value for integers $u_1^*,..., u_n^*$, say. We may assume that these integers are relatively prime, and by applying a suitable unimodular substitution to the variables $u_1,..., u_n$ we may suppose further that $(u_1^*, u_2^*,..., u_n^*) = (1, 0,..., 0)$. Thus, for any $\epsilon > 0$, we may write

$$\pm\wp = a(\bar{L}_1+\alpha)^2+\bar{\wp}(u_2,..., u_n),$$

where

$$0 < a < \epsilon.$$

Here $\bar{L}_1$ is a linear form in $u_1,..., u_n$ with the coefficient of u_1 equal to unity and $\bar{\wp} = \bar{\wp}(u_2,..., u_n)$ is a polynomial of the form

$$\bar{\wp} = \pm\bar{L}_2^2 \pm ... \pm \bar{L}_{n-1}^2 + \bar{L}_n + \bar{c},$$

$\bar{L}_2,..., \bar{L}_n$ being linear forms in $u_2,..., u_n$ of determinant $\pm a^{-\frac{1}{2}}\Delta$. If the coefficients of $\bar{\wp}-\bar{c}$ are not all in a rational ratio, by our inductive hypothesis, there exist integers $u_2,..., u_n$ satisfying

$$|\bar{\wp}| < \tfrac{1}{2}\epsilon.$$

If we now choose an integer u_1 so that

$$|\bar{L}_1+\alpha| \leqslant \tfrac{1}{2},$$

we have, for these values of $u_2,..., u_n$,

$$|\wp| < \tfrac{1}{2}\epsilon+\tfrac{1}{2}\epsilon = \epsilon.$$

Alternatively, if the coefficients of $\bar{\wp}-\bar{c}$ are all in a rational ratio, we apply Theorem 2 or Macbeath's theorem (6) for polynomials in $n-1$ variables. Thus there are integers $u_2,..., u_n$ for which $\bar{\wp} = -\lambda$, where

$$0 \leqslant \lambda \leqslant 2(C_n|\Delta|a^{-\frac{1}{2}})^{2/n},$$

i.e.

$$0 \leqslant a\lambda \leqslant 2a^{1-1/n}(C_n|\Delta|)^{2/n} \tag{43}$$

for some constant C_n depending only on n. For these values of $u_2,..., u_n$ we have

$$\pm\wp = a(u_1+\alpha)^2-\lambda.$$

We can always choose an integer u_1 such that

$$u_1+\alpha = (a^{-1}\lambda)^{\frac{1}{2}}+\theta,$$

where θ is a suitable real number satisfying

$$|\theta| \leqslant \tfrac{1}{2}. \tag{44}$$

With this value of u_1 we have

$$\pm\wp = 2(a\lambda)^{\frac{1}{2}}\theta+a\theta^2,$$

and hence

$$|\wp| \leqslant (a\lambda)^{\frac{1}{2}}+\tfrac{1}{4}a = O(a)$$

by (43) and (44). Since $0 < a < \epsilon$ and ϵ is arbitrary, it follows that $|\wp|$ can be made arbitrarily small for integers $u_1,..., u_n$. This completes the proof of the theorem.

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GAUGE-INVARIANT GENERALIZATION OF FIELD THEORIES WITH ASYMMETRIC FUNDAMENTAL TENSOR (II)

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1. In the first part of this paper (**1**)† I introduced a space J_n provided with an asymmetric fundamental tensor g_{ik}, a covariant vector k_i, and a linear connexion Γ^s_{ik} defined in such a way that Γ^s_{ik} itself and all basic tensors derived from the g_{ab}, k_c, Γ^s_{ik}, and their derivatives shall possess the property of gauge-covariance. Explicitly, Γ^s_{ik} was, in effect, defined as follows: let $\underset{0}{\Gamma}{}^s_{ik}$, γ^s_{ik} be the solutions (supposed to exist) of the equations

$$\underset{0}{\Gamma}{}^s_{il}g_{sk}+\underset{0}{\Gamma}{}^s_{lk}g_{is}=g_{ik,l}, \tag{1.1}$$

$$\gamma^s_{il}g_{sk}+\gamma^s_{lk}g_{is}=g_{ik}k_l. \tag{1.2}$$

Then

$$\Gamma^s_{ik}=\underset{0}{\Gamma}{}^s_{ik}-\gamma^s_{ik}. \tag{1.3}$$

$\underset{0}{\Gamma}{}^s_{ik}$ is of course the linear connexion of the usual 'asymmetrical theory', i.e. of an H_n [cf. (**2**)]. The gauge-invariance [cf. § 2] of Γ^s_{ik} follows from (1.1) and (1.2) by inspection.

In this paper further results relating to the gauge-invariant theory are obtained, most of which arise from the fact that the equation (1.2) possesses a simple explicit solution (§ 2). In particular, § 3 provides explicit expressions for the curvature tensor $B_{ikl}{}^s$ and the Einstein tensor (G_{ik}) in terms of the corresponding tensors in H_n. The question of the existence of gauge-invariant tensors independent of the vector k_i is considered in § 4. Any such tensor naturally defines a *conformal tensor* in H_n. In § 5, I deal with Bianchi's identity and some of its concomitants. Finally, in § 6, I consider a certain set of field equations in the spherically symmetric case, without, however, being able to obtain any 'non-trivial' solutions of these equations; and the possible physical significance of such field equations is briefly discussed in § 7.

2. The explicit solution of (1.2) can be obtained as follows. Write $g_{\underline{ik}}=a_{ik}$, $g_{\underset{\vee}{ik}}=\epsilon\phi_{ik}$ and suppose ϵ to be 'small'. If one then inserts

† The present notation and terminology follow that of (**1**) throughout.

$a_{ik}+\epsilon\phi_{ik}$ for g_{ik} and $\sum_{m=0}^{\infty} \underset{m}{\gamma^s_{ik}}\epsilon^m$ for γ^s_{ik} in (1.2) and requires the resulting equation to be satisfied identically for all values of ϵ, one obtains a set of equations from which $\underset{0}{\gamma^s_{ik}}$, $\underset{1}{\gamma^s_{ik}}$,... can be determined in turn. All $\underset{m}{\gamma^s_{ik}}$ ($m \geqslant 2$) turn out to vanish identically. The *exact* solution of (1.2) is then

$$\gamma^s_{ik} = \tfrac{1}{2}(\delta^s_i g_{tk} k^t + \delta^s_k g_{it} k^t - g_{ik} k^s), \tag{2.1}$$

where

$$k^s = a^{st} k_t, \tag{2.2}$$

the tensor a^{st} being reciprocal to a_{ik}. The symmetrical part of this is

$$\underset{\wedge}{\gamma^s_{ik}} = \tfrac{1}{2}(\delta^s_i k_k + \delta^s_k k_i - a_{ik} k^s), \tag{2.3}$$

and its skew-symmetrical part

$$\underset{\vee}{\gamma^s_{ik}} = \tfrac{1}{2}(\delta^s_k s_i - \delta^s_i s_k - g_{ik} \underset{\vee}{k^s}), \tag{2.4}$$

where

$$s_i = g_{it} \underset{\vee}{k^t}. \tag{2.5}$$

3. (*a*) If one uses (1.3) in the expression (**1**) (1.9) defining the curvature tensor $B_{ikl}{}^s$, one has immediately

$$B_{ikl}{}^s = \underset{0}{B_{ikl}{}^s} + 2(\underset{+0}{\gamma^s_{i[k:l]}} + \gamma^t_{i[l}\gamma^s_{t|k]}). \tag{3.1}$$

Symbols which are provided with a subscript 0 beneath their kernel symbols refer to the H_n whose linear connexion is $\underset{0}{\Gamma^s_{ik}}$, and subscripts following a *colon* denote covariant differentiation with respect to this connexion in the usual way. When the explicit expression (2.1) for γ^s_{ik} is introduced on the right-hand side of (3.1), care must be exercised with regard to symbols such as $\underset{0}{\delta^s_{k:l}}$ or $\underset{+0}{g_{ik:l}}$ which, of course, do not vanish in general. One then obtains

$$\begin{aligned} B_{ikl}{}^s = \underset{0}{B_{ikl}{}^s} &+ \delta^s_i g_{t[k} k^t_{:l]} + g_{it}\delta^s_{[k} k^t_{:l]} - g_{t[k} k^s_{:l]} - \\ &- \delta^s_i g_{tr} \underset{0}{\Gamma^r_{kl}} k^t + g_{it} \underset{0}{\Gamma^s_{kl}} k^t + g_{it} \underset{0}{\Gamma^t_{kl}} k^s + \\ &+ g_{it}\delta^s_{[k} k_{l]} k^t + \tfrac{1}{2} g_{t[k}\delta^s_{l]} k_t k^t + \tfrac{1}{2} g_{it} g_{r[k}\delta^s_{l]} k^t k^r - \tfrac{1}{2} g_{t[k} g_{it]} k^t k^s + \tfrac{1}{2} g_{it} \underset{\vee}{g_{kl}} k^t k^s. \end{aligned} \tag{3.2}$$

This can be given a somewhat more compact form if one introduces the tensor

$$K^a_{\cdot b} = k^a_{:b} + \tfrac{1}{2} k^a k^c g_{cb}. \tag{3.3}$$

Then, for instance, the fourth and eleventh terms on the right-hand side of (3.2) are simply $-g_{t[k} K^s_{\cdot l]}$. Proceeding in this way one obtains

after lengthy manipulation

$$B_{ikl\cdot}{}^{s} = \underset{0}{B}_{ikl\cdot}{}^{s} + g_{il}\delta^{s}_{[k}K^{t}_{\cdot l]} - g_{t[k}K^{s}_{\cdot l]} + \delta^{s}_{i}g_{t[k}K^{t}_{\cdot l]} + \\ + \tfrac{1}{2}g_{i[k}\delta^{s}_{l]}k_{t}k^{t} + 3g_{il}k^{r}\delta^{s}_{[r}(\underset{0}{\Gamma}{}^{t}_{kl]} + \tfrac{1}{2}g_{kl]}k^{t}) + \\ + (\delta^{s}_{r}g_{il} - \delta^{s}_{i}g_{tr})\underset{0}{\Gamma}{}^{r}_{\underset{\vee}{kl}}k^{t}. \qquad (3.4)$$

(*b*) The gauge-invariant hermitian Einstein tensor $`G_{ik}$ [cf. (**1**) (4.2)] can be dealt with in the same way. Thus one easily finds that

$$`G_{ik} = \underset{0}{`G}_{ik} + \gamma^{s}_{\underset{+-}{i\,k:s}} - \tfrac{1}{4}n(k_{\underset{+}{i:k}} + k_{\underset{-}{k:i}}) + \gamma^{s}_{it}\gamma^{t}_{sk} - \tfrac{1}{2}n\gamma^{s}_{ik}k_{s}. \qquad (3.5)$$

It will suffice to quote the final result which takes on a particularly simple form if constant use is made of the vector s_i defined by (2.5); thus

$$`G_{ik} = \underset{0}{`G}_{ik} - \tfrac{1}{2}(\tfrac{1}{2}n-1)(k_{\underset{+}{i:k}} + k_{\underset{-}{k:i}} + k_{i}k_{k}) - \tfrac{1}{2}g_{ik}(k^{t}_{\underset{-}{t:t}} - (\tfrac{1}{2}n-1)k_{t}k^{t}) + \\ + s_{(i,k)} + k_{(i}\underset{0}{\Gamma}{}^{t}_{k)t} + s_{(i}\underset{0}{\Gamma}{}^{t}_{k)t} - \tfrac{1}{2}(\tfrac{1}{2}n-1)s_{i}s_{k}. \qquad (3.6)$$

When $g_{\underset{\vee}{ik}} = 0$, this is seen at a glance to reduce correctly to the symmetrical part $G_{(ik)}$ of the corresponding tensor of Weyl's theory [cf. (**1**) (1.5)].

4. (*a*) In Weyl's theory [cf. (**1**) § 1 *b*] let

$$*B_{ikl\cdot}{}^{s} = B_{ikl\cdot}{}^{s} - \tfrac{1}{2}\delta^{s}_{i}F_{kl}, \qquad *G_{ik} = *B_{iks\cdot}{}^{s}, \qquad *G = g^{ik}*G_{ik}. \qquad (4.1)$$

Then the conformal curvature tensor

$$\hat{C}_{ikl\cdot}{}^{s} = *B_{ikl\cdot}{}^{s} + \frac{2}{n-2}(*G_{i[l}\delta^{s}_{k]} - *G^{s}_{[l}g_{k]i}) + \frac{2}{(n-1)(n-2)}g_{i[k}\delta^{s}_{l]}*G \qquad (4.2)$$

does not involve the vector k_i at all, and it is gauge-invariant. The corresponding tensor in a Riemann space V_n is therefore invariant with respect to conformal transformations of the metrical tensor, i.e. it is a conformal tensor. The question therefore arises in the first place whether there exists a gauge-invariant tensor in J_n which (i) is independent of k_i, (ii) is constructed from $B_{ikl\cdot}{}^{s}$ and its concomitants, and (iii) reduces to $C_{ikl\cdot}{}^{s}$ when $g_{\underset{\vee}{ab}} = 0$. To such a tensor there would then correspond a conformal curvature tensor in H_n. I consider it unlikely that such a tensor exists. For the condition that the required tensor be a concomitant of $B_{ikl\cdot}{}^{s}$ excludes artificial, that is trivial, tensors such as the conformal curvature tensor of the W_n whose fundamental tensor and vector are g_{ik} and k_i respectively. On the other hand, if we contemplate (3.2), it does not seem possible to eliminate for instance the

term $g_{il}\overset{0}{\underset{\vee}{\Gamma}}{}^{l}_{kl}k^s$ by drawing upon the concomitants of $B_{ikl}{}^{s}$ such as $\tilde{B}_{ikl}{}^{s}$, $B_{iks}{}^{s}$, $g^{ik}B_{iks}{}^{s}$, etc.

(*b*) There exist, however, in J_n certain non-trivial gauge-invariant tensors of another kind not involving k_i to which there correspond conformal tensors in H_n. These tensors are peculiar to the asymmetrical theories in that they vanish identically when $g_{\underset{\vee}{ik}} = 0$. Thus one has from (2.4) that

$$g_{\underset{\vee}{ls}}\gamma^{s}_{\underset{\vee}{ik}} = -\tfrac{3}{2}g_{(ik}s_{l)}. \tag{4.3}$$

Since

$$\gamma^{s}_{\underset{\vee}{is}} = \gamma_i = \tfrac{1}{2}(n-2)s_i, \tag{4.4}$$

one thus has

$$g_{\underset{\vee}{ls}}\gamma^{s}_{\underset{\vee}{ik}} + \frac{3}{n-2}g_{(ik}\gamma_{l)} = 0. \tag{4.5}$$

It follows that the tensor

$$g_{\underset{\vee}{ls}}\Gamma^{s}_{\underset{\vee}{ik}} + \frac{3}{n-2}g_{(ik}\Gamma_{l)} \tag{4.6}$$

does not involve k_i, whilst its gauge-weight is $+1$. Let c^{ik} stand for a contravariant (algebraic) concomitant of the fundamental tensor, which will be of gauge weight -1. (Possible examples are g^{ik}, a^{ik}, $g^{ik}_{\vee}$ or, when n is even and $\det g_{\underset{\vee}{ab}} \neq 0$, the tensor reciprocal to $g_{\underset{\vee}{ik}}$, and so on.)

Then the tensor

$$p^{l}_{ik} = c^{jl}\left(g_{\underset{\vee}{ls}}\Gamma^{s}_{\underset{\vee}{ik}} + \frac{3}{n-2}g_{(ik}\Gamma_{l)}\right) \tag{4.7}$$

is gauge-invariant and independent of k_i. The corresponding quantity in H_n is therefore a conformal tensor.

It may be noted in parenthesis that in H_n one can evidently define a conformal connexion as was done for a V_n by Thomas (**3**).

5. From the definition of $B_{ikl}{}^{s}$ one obtains by direct computation the Bianchi identity

$$B_{\underset{+++}{ikl}}{}^{s}{}_{;j} + B_{\underset{+-+}{ijk}}{}^{s}{}_{;l} + B_{\underset{+--}{ilj}}{}^{s}{}_{;k} = 0. \tag{5.1}$$

Write† $G_{ik} = B_{\underset{++-}{iks}}{}^{s}$. Setting $k = s$, one then has

$$B_{\underset{+--}{ilj}}{}^{s}{}_{;s} = G_{\underset{++}{il};j} - G_{\underset{+-}{ij};l}. \tag{5.2}$$

If (5.2) be multiplied throughout by g^{ij}, one gets, since $g^{ij}_{\underset{-}{+}\,;s} = 0$,

$$(g^{ij}B_{\underset{-}{ijl}}{}^{s}){}_{;s} + g^{ij}G_{\underset{++}{il};j} = G_{;l}, \tag{5.3}$$

where

$$G = g^{ij}G_{ij}. \tag{5.4}$$

Now write

$$g_{sj}B_{ikl}{}^{s} = B_{iklj}. \tag{5.5}$$

† The tensor denoted by G_{ik} in (**1**) § 5 does not occur in this paper.

Then, following exactly the procedure of Einstein (4), one gets here in the gauge-covariant case

$$B_{iklj} = -\tilde{B}_{jkli} + g_{ij} F_{kl}, \tag{5.6}$$

where the tilde as usual indicates transposition of all the Γ^{c}_{ab} and g_{mn}. Therefore

$$(g^{ij} B_{ij\underset{-}{l}}{}^{\underset{+}{s}})_{;s} = g^{st}(g^{ij} B_{ij\underset{--}{l}\,l})_{;s} = g^{st}[g^{ij}(\tilde{B}_{\underset{--}{l}\,ij} + g_{\underset{-}{l}\underset{-}{i}} F_{\underset{-}{j}\underset{-}{l}})]_{;s}$$

$$= g^{st}(\tilde{G}_{\underset{--}{l}\,\underset{--}{l}} + F_{\underset{--}{l}\,\underset{--}{l}})_{;s}. \tag{5.7}$$

For the sake of simplicity the field will now be subjected to the condition

$$\Gamma_i = 0. \tag{5.8}$$

Then

$$G_{ik} = {}'G_{ik} - \tfrac{1}{4} n F_{ik}, \quad \tilde{G}_{kl} = {}'G_{ik} + \tfrac{1}{4} n F_{ik}, \tag{5.9}$$

where ${}'G_{ik}$ is again the hermitian Einstein tensor (1) (4.2). Because of (1) (4.6), $G = {}'G$. From (5.3), (5.7), (5.9) one now has

$$g^{st}[{}'G_{\underset{++}{s}\,\underset{-}{l}\underset{-}{j}} + {}'G_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}} + \tfrac{1}{4} n(F_{\underset{++}{l}\,\underset{+}{s}\underset{-}{j}} + F_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}}) - F_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}}] = {}'G_{\underset{-}{j}}. \tag{5.10}$$

The identity

$$F_{\underset{+-}{s}\,\underset{-}{t}\underset{-}{j}} + F_{\underset{++}{l}\,\underset{+}{s}\underset{-}{j}} + F_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}} = 0 \tag{5.11}$$

is readily confirmed. Again, with (1) (4.6), this gives

$$g^{st} F_{\underset{++}{l}\,\underset{+}{s}\underset{-}{j}} = g^{st} F_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}},$$

whence (5.10) finally becomes

$$g^{st}[{}'G_{\underset{++}{s}\,\underset{-}{l}\underset{-}{j}} + {}'G_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}} + (\tfrac{1}{2} n - 1) F_{\underset{--}{l}\,\underset{-}{t}\underset{-}{s}}] = {}'G_{\underset{-}{j}}. \tag{5.12}$$

6. (*a*) If one now takes $n = 4$, the simplest field equations in this theory are presumably

$${}'G_{ik} = 0, \qquad \Gamma_i = 0. \tag{6.1), (6.2}$$

In view of (5.12) these entail

$$\mathfrak{F}^{il}{}_{,l} + \Gamma^{i}_{\underset{\vee}{st}} \mathfrak{F}^{st} = 0 \qquad (\mathfrak{F}^{ik} = \sqrt{(-g)}\, g^{is} g^{kt} F_{st}), \tag{6.3}$$

whilst one has also the equations (1) (4.6), viz.

$$g^{ik}_{\vee} F_{ik} = 0. \tag{6.4}$$

The corresponding equations of Weyl's theory,

$$G_{(ik)} = 0, \qquad \mathfrak{F}^{ik}{}_{,k} = 0,$$

do not give physically acceptable results in the static spherically symmetric case [cf. (1) § 1*b* and § 7 below]. It would therefore be of interest to obtain the solution of (6.1), (6.2) in this case.

(*b*) Taking spherical polar coordinates I shall suppose that of the k_i

only k_4 ($= \kappa(r)$, say) does not vanish. The scheme for the g_{ik} is (**5**)

$$g_{ik} = \begin{pmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma, f, w$ are functions of r only. Now of the F_{ik} only $F_{14} \neq 0$. Hence (6.4) requires that $g^{14}_{\vee} = 0$. Consequently $w = 0$, and the only non-vanishing skew component of g_{ik} is $g_{\underset{\vee}{23}}$. This in turn implies that the vector s_i vanishes. But in view of (4.4) equation (6.2) then becomes

$$\Gamma^i_{\underset{0}{jk}} = 0. \tag{6.5}$$

From (3.6) equation (6.1) now gives

$$`G_{\underset{0}{ik}} = \tfrac{1}{2}(k_{\underset{+}{i:k}} + k_{\underset{-}{k:i}} + k_i k_k) + \tfrac{1}{2} g_{ik}(k^l_{\underset{-}{:l}} - k_l k^l). \tag{6.6}$$

Since $g_{\underset{\vee}{14}} = 0$, one has $`G_{\underset{0\vee}{14}} = 0$ also,† whence from (6.6)

$$k_{\underset{+}{1:4}} + k_{\underset{-}{4:1}} = 0. \tag{6.7}$$

If a prime denotes differentiation with respect to r, (6.7) yields

$$\kappa' - 2\Gamma^4_{14}\kappa = 0. \tag{6.8}$$

Since $\Gamma^4_{14} = \gamma'/2\gamma$, this gives at once

$$\kappa = a\gamma, \tag{6.9}$$

where a is a constant of integration. It is easily confirmed that

$$k^l_{\underset{-}{:l}} = 0, \qquad k^l k_l = \kappa^2/\gamma = a^2\gamma. \tag{6.10}$$

The remaining equations of (6.6) now become

$$`G_{\underset{0}{11}} = -\tfrac{1}{2}a^2\gamma g_{11}, \quad `G_{\underset{0}{22}} = -\tfrac{1}{2}a^2\gamma g_{22}, \quad `G_{\underset{0}{44}} = 0,$$
$$`G_{\underset{0\vee}{23}} = -\tfrac{1}{2}a^2\gamma g_{\underset{\vee}{23}}. \tag{6.11}$$

The third of these, i.e. $`G_{\underset{0}{44}} = 0$, is equation (1.3) of Wyman's paper (**5**), and this can be integrated directly to give

$$\gamma' = 2m[\alpha\gamma/(f^2+\beta^2)]^{\frac{1}{2}}, \tag{6.12}$$

where m is a constant of integration. It may be noticed that equation (6.3) under the present conditions reduces simply to

$$(\mathfrak{F}^{14})' = 0, \tag{6.13}$$

which is readily seen to be in agreement with (6.12).

† Cf. the paper by Bonnor (**6**) in which the explicit expressions for Γ^a_{ik} and $`G_{ik}$ ($= -R_{ik}$) are listed.

Beyond this point I have unfortunately not been able to make any progress with the exact solution of these equations. It may however be useful to indicate briefly how the relevant equations of Wyman's paper are modified (apart, of course, from having to equate to zero the constant c which occurs in them). The left-hand members of Wyman's equations (1.2) and (1.8) are now equated to $\frac{1}{2}a^2\alpha\gamma$ and $a^2\alpha\gamma$ respectively (instead of to zero), whilst equations (1.6) and (1.9) remain unchanged. Unfortunately the presence of the additional terms vitiates the previous method of solution.

7. Although the present investigation has been intended to be mainly of a mathematical nature, the question arises as to the extent to which the equations developed above might describe physical fields, amongst which the electromagnetic field would certainly be expected to find a place. The mere occurrence of a single bivector in a set of field equations naturally suggests the identification of the electromagnetic field tensor with it, or with one of its algebraic concomitants. In an H_n this already implies a wide *a priori* choice [cf. Tonnelat (**7**)]. In a J_n however there is the additional bivector F_{ik}, so that the freedom of choice is greatly widened.

Under such circumstances it is usual to search for possible identifications by considering special cases. The most prominent amongst these are (i) linear approximations, (ii) exact (static) solutions in the case of spherical symmetry. In an H_n, and more so in a J_n, the linear approximation is, however, unsatisfactory for various reasons. Thus, non-linearity is essential to a unitary theory; peculiar restrictions may appear even in the quasi-linear approximations which are absent in the linear case, e.g. Buchdahl (**8**); and in a J_n the field equations may imply directly certain restrictions which are of an essentially non-linear character. If the field equations (6.1), (6.2) above be accepted, then equation (6.4) is a good example of such a non-linear restriction, which in the spherically symmetric case requires that g_{14} be zero (unless $\kappa' = 0$, which is trivial).

As regards the general exact solution in the spherically symmetric case, this is unfortunately not available, as pointed out at the end of the previous section. However, even without it the behaviour of the solution at infinity leads to the conclusion that (6.1), (6.2) at any rate are not likely to describe physical fields. Thus, as $r \to \infty$, if

$$\alpha = 1+O(r^{-1}), \quad \beta = r^2, \quad \gamma = 1+O(r^{-1}), \quad f = \text{constant}+O(r^{-1}),$$
$$A \sim 2r^{-1}, \qquad B \sim \text{constant} \times r^{-3}, \tag{7.1}$$

then Wyman's equation (1.2) as amended above leads to

$$O(r^{-3}) = \tfrac{1}{2}a^2: \tag{7.2}$$

that is, a must vanish, in which case one merely returns to the solution of the strong equations of Einstein's original theory. The conditions (7.1) of course imply the assumption that J_n is galilean at infinity, meaning thereby that the $g_{\underline{ik}}$ tend asymptotically to galilean values and the $g_{\underset{\vee}{ik}}$ tend asymptotically to zero as r tends to infinity. Now k_i and $g_{\underset{\vee}{ef}}$ are *a priori* independent of each other (and this may be thought to be an additional defect of theories such as this), and, when $g_{\underset{\vee}{23}} = 0$, one has the exact spherically symmetric solution of Weyl's theory, viz.

$$\alpha\gamma = (b+\tfrac{1}{4}a^2r^2)^{-1}, \quad \beta = r^2, \quad \gamma = \frac{1}{b} - \frac{2m}{r}\sqrt{\left(1+\frac{a^2}{4b}r^2\right)}, \quad \kappa = a\gamma, \tag{7.3}$$

where a, b, m are constants of integration. If W_4 is to be galilean at infinity, one must have $a = 0$, in harmony with the previous result. As long therefore as one requires J_n to be galilean at infinity, it is likely that the equations (6.1), (6.2) have no acceptable solutions.

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ON CLAUSEN'S HYPERGEOMETRIC IDENTITY

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1. THIS note has had the advantage of much helpful criticism from Professor J. L. Burchnall. His contributions are indicated by [J.L.B.].

Clausen (**3**) [quoted by Bailey (**1**) 86 (4)] gave the identity

$$\{F(a,b;\,a+b+\tfrac{1}{2};\,x)\}^2 = {}_3F_2(2a,\,a+b,\,2b;\,a+b+\tfrac{1}{2},\,2a+2b;\,x), \quad (1)$$

where F is the elementary ${}_2F_1$. In this note I consider (1) and some associated identities.

In the first place let us remark that the differential equation satisfied by

$$Y = \{F(a,b;\,c;\,x)\}^2$$

for unrestricted c can be obtained in the form

$$(1-x)\{\delta(\delta+c-1)(\delta+2c-2)-x(\delta+2a)(\delta+a+b)(\delta+2b)\}Y+$$
$$+(2h-1)x(h\delta+2ab)Y = 0, \quad (2)$$

where $h = a+b-c+1$.

For a proof [J.L.B.] we can use the algebraic identity

$$\Theta(a,b) \equiv (\delta+2a)(\delta+a+b)(\delta+2b)$$
$$= (\theta+3\phi+2a+2b)(\theta+a)(\theta+b)+$$
$$+(3\theta+\phi+2a+2b)(\phi+a)(\phi+b), \quad (3)$$

where $\delta = \theta+\phi$. This is readily verifiable, for, putting $\phi = -b$, we get $(\theta+2a-b)(\theta+a)(\theta+b)$ on the two sides. Thus, by symmetry, the expressions on the two sides differ by a multiple of

$$(\theta+a)(\theta+b)(\phi+a)(\phi+b);$$

the degree is excessive, and hence (3) is an identity. Operate with this identity on $F^2 \equiv F.F$ with the convention that θ is δ operating only on the first factor, ϕ only on the second factor. Then

$$\Theta(a,b)F^2 = \theta(\theta+a)(\theta+b)F.F+(\theta+a)(\theta+b)F.(3\phi+2a+2b)F+$$
$$+F.\phi(\phi+a)(\phi+b)F+(3\theta+2a+2b)F.(\phi+a)(\phi+b)F.$$

Thus, using the differential equation for F, we have

$$x\Theta(a,b)F^2 = (\theta-1)\theta(\theta+c-1)F.F+\theta(\theta+c-1)F.(3\phi+2a+2b)F+$$
$$+F.(\phi-1)\phi(\phi+c-1)F+(3\theta+2a+2b)F.\phi(\phi+c-1)F$$
$$= \{(\theta+3\phi+2a+2b-1)\theta(\theta+c-1)+$$
$$+(3\theta+\phi+2a+2b-1)\phi(\phi+c-1)\}F^2.$$

Hence

$$\{\Theta(0,c-1)-x\Theta(a,b)\}F^2 = (1-2h)\{\theta(\theta+c-1)+\phi(\phi+c-1)\}F^2.$$

Now

$$(1-x)\theta(\theta+c-1) = \theta(\theta+c-1)-x(\theta+a)(\theta+b)+x(h\theta+ab).$$

Thus

$$(1-x)\{\Theta(0,c-1)-x\Theta(a,b)\} = (1-2h)x(h\delta+2ab)F^2,$$

which is (2).

When $h = \frac{1}{2}$, i.e. $c = a+b+\frac{1}{2}$, the equation (2) reduces to the third-order hypergeometric equation

$$\delta(\delta+a+b-\tfrac{1}{2})(\delta+2a+2b-1)Y = x(\delta+2a)(\delta+a+b)(\delta+2b)Y, \quad (4)$$

which has as a solution the right-hand member of (1), and we are led to Clausen's identity (1).

But we can go further. Write

$$y_1 = F(a,b;c;x), \qquad y_2 = x^{1-c}F(a-c+1,b-c+1;2-c;x)$$

for the two distinct solutions of

$$\delta(\delta+c-1)y = x(\delta+a)(\delta+b)y.$$

Then y_1^2, y_1y_2, y_2^2 are three distinct solutions of (2). When $c = a+b+\frac{1}{2}$, we identify y_1y_2 as the solution of (3) led by $x^{\frac{1}{2}-a-b}$. This is

$$x^{\frac{1}{2}-a-b}\,{}_3F_2(\tfrac{1}{2},a-b+\tfrac{1}{2},b-a+\tfrac{1}{2};a+b+\tfrac{1}{2},\tfrac{3}{2}-a-b;x).$$

We can thus associate with Clausen's identity (1) the further identity

$$F(a,b;a+b+\tfrac{1}{2};x)F(\tfrac{1}{2}-a,\tfrac{1}{2}-b;\tfrac{3}{2}-a-b;x)$$
$$= {}_3F_2(\tfrac{1}{2},a-b+\tfrac{1}{2},b-a+\tfrac{1}{2};a+b+\tfrac{1}{2},\tfrac{3}{2}-a-b;x). \quad (5)$$

The identification of y_2^2 with the third solution of (4) gives only a version of (1).

2. We can extend Clausen's identity for unrestricted c by the expansion

$$\{F(a,b;c;x)\}^2$$
$$= \sum_{n=0}^{\infty} \frac{(2a)_n(2b)_n(c-\frac{1}{2})_n}{n!(c)_n(2c-1)_n}\,{}_4F_3\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n,\ \frac{1}{2},\ a+b-c+\frac{1}{2};\\ a+\frac{1}{2},\ b+\frac{1}{2},\ \frac{3}{2}-c-n\end{matrix}\right]x^n, \quad (6)$$

where the ${}_4F_3$ is of unit argument, which, by convention, is omitted from the notation. This can be quickly obtained as follows [J.L.B.]. We have [(**2**) 70 (50)]

$$\{F(a,b;c;x)\}^2$$

$$=\sum_{r=0}^{\infty}\frac{(\frac{1}{2})_r(a)_r(b)_r(c-a)_r(c-b)_r}{r!(c)_r(c)_{2r}(c+r-\frac{1}{2})_r}x^{2r}F(2a+2r,2b+2r;2c+4r;x).$$

On the right the coefficient of x^n is

$$\sum_{r=0}^{[\frac{1}{2}n]}\frac{(\frac{1}{2})_r(a)_r(b)_r(c-a)_r(c-b)_r}{r!(c)_r(c)_{2r}(c+r-\frac{1}{2})_r}\cdot\frac{(2a+2r)_{n-2r}(2b+2r)_{n-2r}}{(n-2r)!\,(2c+4r)_{n-2r}},$$

which can be rewritten as

$$\frac{(2a)_n(2b)_n}{n!(2c)_n}\,{}_7F_6\left[\begin{matrix}c-\frac{1}{2},\ \frac{1}{2}c+\frac{3}{4},\ \frac{1}{2},\ c-a,\ c-b,\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n\\ \frac{1}{2}c-\frac{1}{4},\ c,\ a+\frac{1}{2},\ b+\frac{1}{2},\ c+\frac{1}{2}n+\frac{1}{2},\ c+\frac{1}{2}n\end{matrix}\right].$$

By Whipple's formula [(**1**) 25 (4)] the ${}_7F_6$ reduces to

$$\frac{(c+\frac{1}{2})_m(c-e)_m}{(c)_{\frac{1}{2}m}(c-e-\frac{1}{2})_m}, \tag{7}$$

where m, $-e=\frac{1}{2}n$, $\frac{1}{2}(n-1)$ in one or other order, m being the integer. It is soon seen that in either case (7) can be written

$$\frac{(c+\frac{1}{2}n-\frac{1}{2})(c-\frac{1}{2})_n}{(c-\frac{1}{2})(c)_n},$$

and the proof of (6) is complete.

3. We get some identities in terminated ${}_4F_3$ by equating coefficients of x^n on the two sides of the foregoing identities. We have on rearrangement, for any $a, b, c, \alpha, \beta, \gamma$,

$$F(a,b;c;x)F(\alpha,\beta;\gamma;x)$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!\,(c)_n}\,{}_4F_3\left[\begin{matrix}-n,\ \alpha,\ \beta,\ 1-c-n;\\ \gamma,\ 1-a-n,\ 1-b-n\end{matrix}\right]x^n. \tag{8}$$

Thus, writing a, b, c for α, β, γ, we get from (5) the identity

$${}_4F_3\left[\begin{matrix}-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n,\ \frac{1}{2},\ a+b-c+\frac{1}{2};\\ a+\frac{1}{2},\ b+\frac{1}{2},\ \frac{3}{2}-c-n\end{matrix}\right]$$

$$=\frac{(a)_n(b)_n(2c-1)_n}{(2a)_n(2b)_n(c-\frac{1}{2})_n}\,{}_4F_3\left[\begin{matrix}-n,\ a,\ b,\ 1-c-n;\\ c,\ 1-a-n,\ 1-b-n\end{matrix}\right]. \tag{9}$$

This is, in fact, an alternative statement of (5) since (5) is deducible from (9). The similar restatement of Clausen's identity (1) is

$$ {}_4F_3\left[\begin{matrix} -n,\ a,\ b,\ \frac{1}{2}-a-b-n \\ a+b+\frac{1}{2},\ 1-a-n,\ 1-b-n \end{matrix}\right] = \frac{(2a)_n(2b)_n(c-\frac{1}{2})_n}{(a)_n(b)_n(2c-1)_n} \qquad (10) $$

and of (5) is

$$ {}_4F_3\left[\begin{matrix} -n,\ a,\ b,\ a+b-n-\frac{1}{2} \\ a-n+\frac{1}{2},\ b-n+\frac{1}{2},\ a+b+\frac{1}{2} \end{matrix}\right] = \frac{(\frac{1}{2})_n(a-b+\frac{1}{2})_n(b-a+\frac{1}{2})_n}{(\frac{1}{2}-a)_n(\frac{1}{2}-b)_n(a+b+\frac{1}{2})_n}. \qquad (11) $$

Of these (9) is deducible [J.L.B.] from a result of Whipple's, quoted as § 4.7 (1) of (**1**) 33, when we follow it with § 7.2 (1) of (**1**) 56.

Since

$$ F(a+\tfrac{1}{2}, b+\tfrac{1}{2}; a+b+\tfrac{1}{2}; x) = (1-x)^{-\frac{1}{2}}F(a, b; a+b+\tfrac{1}{2}; x) $$

by Euler's identity, we have [J.L.B.]

$$ \{F(a+\tfrac{1}{2}, b+\tfrac{1}{2}; a+b+\tfrac{1}{2}; x)\}^2 = (1-x)^{-1}\{F(a, b; a+b+\tfrac{1}{2}; x)\}^2 $$
$$ = (1-x)^{-1}{}_3F_2(2a, a+b, 2b; a+b+\tfrac{1}{2}, 2a+2b; x). $$

Expanding the left-hand member by (6) with appropriate change of parameters and picking out the coefficients of x^n on the two sides we get

$$ {}_3F_2\left[\begin{matrix} 2a,\ a+b,\ 2b; \\ a+b+\frac{1}{2},\ 2a+2b \end{matrix}\right] \quad \text{to } n+1 \text{ terms} $$
$$ = \frac{(2a+1)_n(2b+1)_n(a+b)_n}{n!\,(a+b+\frac{1}{2})_n(2a+2b)_n}\,{}_4F_3\left[\begin{matrix} -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n,\ \frac{1}{2},\ 1 \\ a+1,\ b+1,\ 1-a-b-n \end{matrix}\right], \qquad (12) $$

a partial-summation formula that may be new.

4. The differential equation (2) satisfied by $Y = F^2$ is of rank two, in the sense that it has a form

$$ f_0(\delta)Y + xf_1(\delta)Y + x^2f_2(\delta)Y = 0 $$

in three terms. This, as we have seen, reduces to an equation of rank one, i.e. to a hypergeometric form

$$ g_0(\delta)Y + xg_1(\delta)Y = 0, $$

when $c = a+b+\frac{1}{2}$. As we have also seen, the equation of rank two has a solution in power series in which a terminated ${}_rF_s$ enters as a factor of the coefficients: this can often occur with equations of rank two.

When, as below, we laboriously form the differential equation satisfied by $\{F(a, b; c; x)\}^3$ we obtain (not unexpectedly) an equation of rank three. Its rank reduces to two when either (i) $c = a+b+\frac{1}{2}$ or (ii) $c = a+b+\frac{1}{3}$.

The case (i) is soon disposed of: we need merely the product

$$F(a,b;\,a+b+\tfrac{1}{2};\,x)\,{}_3F_2(2a,a+b,2b;\,a+b+\tfrac{1}{2},2a+2b;\,x).$$

On expansion, multiplication, and rearrangement on the lines of (8) we get

$$\{F(a,b;\,a+b+\tfrac{1}{2};\,x)\}^3$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!\,(a+b+\frac{1}{2})_n}\,{}_5F_4\left[\begin{matrix}-n,\ 2a,\ a+b,\ 2b,\ \frac{1}{2}-a-b-n;\\ a+b+\frac{1}{2},\ 2a+2b,\ 1-a-n,\ 1-b-n\end{matrix}\right]x^n. \tag{13}$$

5. In case (ii) we find that the differential equation satisfied by

$$Z=\{F(a,b;\,a+b+\tfrac{1}{3};\,x)\}^3$$

is

$$\begin{aligned}(1-x)\{&\delta(\delta+c-1)(\delta+2c-2)(\delta+3c-3)-\\&-x(\delta+3a)(\delta+2a+b)(\delta+a+2b)(\delta+3b)\}Z+\\&+x\{\tfrac{20}{9}\delta^2+10(ab+\tfrac{1}{3}c-\tfrac{3}{9})\delta+9ab(ab+c-\tfrac{4}{9})\}Z=0.\end{aligned} \tag{14}$$

To prove this we use [J.L.B.], in extension of (3) above, the identity

$$\begin{aligned}\Phi(a,b)\equiv{}&(\delta+3a)(\delta+2a+b)(\delta+a+2b)(\delta+3b)\\=\sum\{&(\delta+2a+b)(\delta+a+2b)+(2\phi+a+b)(\delta+\psi+2a+2b)+\\&+(2\psi+a+b)(\delta+\phi+2a+2b)+(\phi+a)(\psi+b)+\\&+(\psi+a)(\phi+b)\}(\theta+a)(\theta+b),\end{aligned} \tag{15}$$

where $\delta=\theta+\phi+\psi$ and the summation is over θ, ϕ, ψ. To prove this put $\psi=-b$ and reduce on the right; we then require

$$\begin{aligned}&(\delta_1+3a-b)(\delta_1+2a)(\delta_1+a+b)(\delta_1+2b)\\&\qquad=(\delta_1+3a-b)\sum(\theta+3\phi+2a+2b)(\theta+a)(\theta+b),\end{aligned}$$

where $\delta_1=\theta+\phi$ and the summation is now over θ, ϕ. This is true by (3), and so, by symmetry, the difference of the expressions on the two sides of (15) is divisible by

$$\prod_{\theta,\phi,\psi}(\theta+a)(\phi+a),$$

which is of excessive degree, and thus (15) is established. We operate with Φ on $F^3=F.F.F$, where θ, ϕ, ψ operate separately on successive

factors, getting

$$\Phi(0,c-1)F^3$$
$$= \sum\{(\delta+c-1)(\delta+2c-2)+(2\phi+c-1)(\delta+\psi+2c-2)+$$
$$+(2\psi+c-1)(\delta+\phi+2c-2)+\phi(\psi+c-1)+\psi(\phi+c-1)\}\theta(\theta+c-1)F^3,$$
$$x\Phi(a,b)F^3$$
$$= \sum\{(\delta+2a+b-1)(\delta+a+2b-1)+(2\phi+a+b)(\delta+\psi+2a+2b-2)+$$
$$+(2\psi+a+b)(\delta+\phi+2a+2b-1)+(\phi+a)(\psi+b)+$$
$$+(\psi+a)(\phi+b)\}F^2\theta(\theta+c-1)F.$$

Then

$$\{\Phi(0,c-1)-x\Phi(a,b)\}F^3$$
$$= \sum\{(11k-7)\delta-(6k-4)\theta+(12k-7)c-$$
$$-(6k^2+5k-5)-3ab\}\theta(\theta+c-1)F^3,$$

where $k = c-a-b$. Now

$$(1-x)\theta(\theta+c-1) = x\{(1-k)\theta+ab\},$$
$$(1-x)\theta^2(\theta+c-1) = x\theta\{(1-k)\theta+ab\}+x(\theta+a)(\theta+b),$$
$$(1-x)\delta\theta(\theta+c-1) = x\delta\{(1-k)\theta+ab\}+x(\theta+a)(\theta+b).$$

Thus

$$(1-x)\{\Phi(0,c-1)-x\Phi(a,b)\}F^3$$
$$= x\sum[\{(11k-7)\delta-(6k-4)\theta+(12k-7)c-(6k^2+5k-5)-3ab\}\times$$
$$\times\{(1-k)\theta+ab\}+(5k-3)(\theta+a)(\theta+b)]F^3$$
$$= (6k^2-5k+1)x(\sum\theta^2)F^3+[(11k-7)\delta\{(1-k)\delta+3ab\}-$$
$$-\{(3k-1)ab+(12k^2-24k+10)c-(6k^3-6k^2-7k+5)\}\delta-$$
$$-3ab(3ab+3c+6k^2-1)]F^3. \quad (16)$$

This equation is of rank two if and only if the coefficient of $\sum\theta^2$ vanishes: that is when $k = \frac{1}{2}, \frac{1}{3}$. Substituting the latter value we get the differential equation (14).

6. To obtain the solution in series we operate on (14) with

$$(\delta+c-\tfrac{7}{3})(\delta+3c-2).$$

The differential equation can then be given the form

$$[\{(\delta+c-\tfrac{7}{3})(\delta+2c-2)(\delta+3c-3)-$$
$$-x(\delta+2a+b-1)(\delta+a+2b-1)(\delta+3c-1)\}\times$$
$$\times\{\delta(\delta+c-1)(\delta+3c-2)-x(\delta+3a)(\delta+3b)(\delta+c-\tfrac{1}{3})\}-$$
$$-\tfrac{2}{3}abx\delta(\delta-1)(4\delta+6c-5)]Z = 0. \quad (17)$$

This is at once clear as regards the end-terms (the terms in 1 and x^2). For the term in x we need the 'algebraic' identity

$$\begin{aligned}(\delta+c-\tfrac{4}{3})(\delta+3c-1)\{&\delta(\delta+c-1)(\delta+2c-2)(\delta+3c-3)+\\&+(\delta+3a)(\delta+2a+b)(\delta+a+2b)(\delta+3b)-\\&-\tfrac{20}{9}\delta^2-10(ab+\tfrac{1}{3}c-\tfrac{2}{9})\delta-9ab(ab+c-\tfrac{4}{9})\}\\=(\delta+3a)(\delta+3b)&(\delta+c-\tfrac{1}{3})(\delta+c-\tfrac{4}{3})(\delta+2c-1)(\delta+3c-2)+\\+\delta(\delta+2a+b-1)&(\delta+a+2b-1)(\delta+c-1)(\delta+3c-1)(\delta+3c-2)+\\&+\tfrac{2}{3}ab\delta(\delta-1)(4\delta+6c-5). \quad (18)\end{aligned}$$

To establish this we can arrange it in powers of ab, remembering that $a+b=c-\frac{1}{3}$. The terms in a^2b^2 disappear at once; the term independent of ab is got by putting $a=c-\frac{1}{3}$, $b=0$. We complete the argument by interpolation: suitable values of δ readily suggest themselves.

Substituting

$$Z=\sum_{n=0}^{\infty}\frac{(3a)_n(3b)_n(c-\frac{1}{3})_n}{n!(c)_n(3c-1)_n}\mu_n x^n$$

into (17), we note first that

$$\begin{aligned}\{\delta(\delta+c-1)(\delta+3c-2)-x(\delta+3a)(\delta+3b)(\delta+c-\tfrac{1}{3})\}Z&\\=\sum_{n=1}^{\infty}\frac{(3a)_n(3b)_n(c-\frac{1}{3})_n}{(n-1)!\,(c)_{n-1}(3c-1)_{n-1}}(\mu_n-\mu_{n-1})x^n.&\end{aligned}$$

Picking out the term in

$$\frac{(3a)_{n-1}(3b)_{n-1}(c-\frac{1}{3})_{n-1}}{(n-1)!\,(c)_{n-1}(c-1)_{n-1}}$$

gives the recurrence relation

$$\begin{aligned}(3a+n-1)(3b+n-1)&(c+n-\tfrac{4}{3})(c+n-\tfrac{7}{3})(2c+n-2)\times\\&\times(3c+n-3)(\mu_n-\mu_{n-1})\\=(n-1)(2a+b+n-2)&(a+2b+n-2)(c+n-2)(3c+n-2)\times\\&\times(3c+n-3)(\mu_{n-1}-\mu_{n-2})+\\&+\tfrac{2}{3}ab(n-1)(n-2)(6c+4n-9)\mu_{n-1}. \quad (19)\end{aligned}$$

I now show that this relation is satisfied by the terminated hypergeometric series of argument unity,

$$H_n={}_9F_8\left[\begin{matrix}c-\frac{2}{3}, \frac{1}{2}c+\frac{2}{3}, & a, & b, & \frac{1}{3}, & -\frac{1}{3}n, & \frac{1}{3}-\frac{1}{3}n, & \frac{2}{3}-\frac{1}{3}n, & 2c+n-1;\\ \frac{1}{2}c-\frac{1}{3}, b+\frac{2}{3}, & a+\frac{2}{3}, & c, & c+\frac{1}{3}n+\frac{1}{3}, & c+\frac{1}{3}n, & c+\frac{1}{3}n-\frac{1}{3}, & -c-n-\frac{4}{3}\end{matrix}\right]. \quad (20)$$

It should be noted that this is well-poised and of the special form discussed by W. N. Bailey in (**1**) [27 (7)].

To prove that (20) satisfies the relation (19) write

$$F_n(x) = x^{\frac{1}{3}c-\frac{1}{3}}{}_8F_7\left[\begin{matrix} c-\frac{2}{3}, & a, & \ldots, & 2c+n-1; \\ & b+\frac{2}{3}, & \ldots, & -c-n+\frac{4}{3} \end{matrix}\; x\right], \tag{21}$$

$$G(x) = x^{\frac{1}{3}c-\frac{1}{3}}{}_8F_7\left[\begin{matrix} c-\frac{2}{3}, & a, & \ldots, & 2c+n-3; \\ & b+\frac{2}{3}, & \ldots, & -c-n+\frac{10}{3} \end{matrix}\; x\right], \tag{22}$$

where the leaders (...) indicate that the omitted parameters are those of the ${}_9F_8$ in (20). Then

$$f(\delta)G = xf(-\delta)G, \tag{23}$$

where

$$f(\delta) \equiv (\delta-\tfrac{1}{2}c+\tfrac{1}{3})(\delta+a-\tfrac{1}{2}c)(\delta+b-\tfrac{1}{2}c)(\delta+\tfrac{1}{2}c-\tfrac{2}{3})\times \\ \times(\delta+\tfrac{1}{2}c+\tfrac{1}{3}n-\tfrac{1}{3})(\delta+\tfrac{1}{2}c+\tfrac{1}{3}n-\tfrac{2}{3})(\delta+\tfrac{1}{2}c+\tfrac{1}{3}n-1)(\delta-\tfrac{3}{2}c-n+\tfrac{8}{3}). \tag{24}$$

In $f(\delta)$ the term in δ^7 is absent. Thus $f(\delta)-f(-\delta)$ has only the terms δ^5, δ^3, δ, and we may therefore write

$$f(\delta)-f(-\delta) = 2\delta\{P(\delta^2-q^2)(\delta^2-r^2)+Q(\delta^2-r^2)(\delta^2-p^2)+ \\ +R(\delta^2-p^2)(\delta^2-q^2)\}, \tag{25}$$

where p, q, r are constants at our disposal. Take

$$p = \tfrac{3}{2}c+n-\tfrac{8}{3}, \quad q = \tfrac{1}{2}c+\tfrac{1}{3}n-\tfrac{1}{3}, \quad r = \tfrac{1}{2}c-\tfrac{1}{3},$$

rewriting (24) as

$$f(\delta) = (\delta+p)(\delta+q)(\delta-r)ff(\delta), \tag{26}$$

where

$$ff(\delta) = (\delta+a-\tfrac{1}{2}c)(\delta+b-\tfrac{1}{2}c)(\delta+\tfrac{1}{2}c-\tfrac{2}{3})(\delta+\tfrac{1}{2}c+\tfrac{1}{3}n-\tfrac{2}{3})(\delta+\tfrac{1}{2}c+\tfrac{1}{3}n-1). \tag{27}$$

From the properties of δ we readily find that

$$\begin{aligned} (\delta^2-p^2)\{\delta^2-(p+1)^2\}G &= (r^2-p^2)\{r^2-(p+1)^2\}F_n, \\ (\delta^2-p^2)(\delta^2-q^2)G &= (r^2-p^2)(r^2-q^2)F_{n-1}, \\ (\delta^2-q^2)\{\delta^2-(q-\tfrac{1}{3})^2\}G &= (r^2-q^2)\{r^2-(q-\tfrac{1}{3})^2\}F_{n-2}, \end{aligned} \tag{28}$$

whence

$$\{(p+1)^2-q^2\}(\delta^2-r^2)(\delta^2-p^2)G \\ = (r^2-p^2)(r^2-q^2)\{r^2-(p+1)^2\}(F_n-F_{n-1}), \tag{29}$$

$$\{p^2-(q-\tfrac{1}{3})^2\}(\delta^2-r^2)(\delta^2-q^2)G \\ = (r^2-p^2)(r^2-q^2)\{r^2-(q-\tfrac{1}{3})^2\}(F_{n-1}-F_{n-2}). \tag{30}$$

Further,

$$\delta F_n = x^{\frac{1}{3}c-\frac{1}{9}}{}_9F_8\left[\begin{matrix} c-\frac{2}{3}, & \frac{1}{2}c+\frac{2}{3}, & a, & \ldots, & 2c+n-1; \\ \frac{1}{2}c-\frac{1}{3}, & b+\frac{2}{3}, & \ldots, & & -c-n-\frac{4}{3} \end{matrix}\; x\right]. \tag{31}$$

Put $x = 1$ in the differential equation (23) and use the identity (25). Then in virtue of (29), (30), (28), (31) we reach

$$Q\frac{(p+1)^2-r^2}{(p+1)^2-q^2}(H_n-H_{n-1}) = P\frac{(q-\frac{1}{3})^2-r^2}{(q-\frac{1}{3})^2-p^2}(H_{n-1}-H_{n-2})+RH_{n-1}. \tag{32}$$

Putting in turn $\delta = -p, q, -r$ in (25) and using (26), we get

$$P = \frac{ff(-p)}{(p+q)(p-r)}, \qquad Q = \frac{ff(q)}{(p+q)(q+r)}, \qquad R = -\frac{ff(-r)}{(p-r)(q+r)}.$$

Completing the calculations (still remembering that $c = a+b+\frac{1}{3}$) and effecting some cancellations we get for H_n precisely the recurrence relation (19). Further $\mu_0 = 1$, $\mu_1 = 1$ since

$$\{F(a,b;c;x)\}^3 = 1+\frac{3ab}{c}x+\ldots,$$

and $H_0 = 1$, $H_1 = 1$ from (20). Thus identically $\mu_n = H_n$, and we have explicitly

$$\{F(a,b;a+b+\tfrac{1}{3};x)\}^3 = \sum_{n=0}^{\infty}\frac{(3a)_n(3b)_n(c-\frac{1}{3})_n}{n!\,(c)_n(3c-1)_n}\times$$

$$\times\,{}_9F_8\left[\begin{matrix} c-\frac{2}{3}, & \frac{1}{2}c+\frac{2}{3}, & a, & b, & \frac{1}{3}, \\ \frac{1}{2}c-\frac{1}{3}, & b+\frac{2}{3}, & a+\frac{2}{3}, & c, & \end{matrix}\right.$$

$$\left.\begin{matrix} -\frac{1}{3}n, & \frac{1}{3}-\frac{1}{3}n, & \frac{2}{3}-\frac{1}{3}n, & 2c+n-1; \\ c+\frac{1}{3}n+\frac{1}{3}, & c+\frac{1}{3}n, & c+\frac{1}{3}n-\frac{1}{3}, & -c-n+\frac{4}{3} \end{matrix}\right]x^n, \tag{33}$$

where $c = a+b+\frac{1}{3}$.

As I have said above the ${}_9F_8$ here is included in the class of ${}_9F_8$ which is the subject of an identity given by Bailey in 1929 and which he very properly claims to be the most advanced of its type. I cannot however discover that this identity has any application to (33) above, but one may remark that in (33) the usual 'parameter of termination' $-n$ appears on the right as the triplet $-\frac{1}{3}n$, $-\frac{1}{3}(n-1)$, $-\frac{1}{3}(n-2)$. Although 'twinning' is common (as mentioned above in § 3), this is the first instance I have seen of a triplet.

I think that (33) is as far as we can usefully push the extension of Clausen's idea: I get nothing presentable from higher powers of ${}_2F_1(a, b; c; x)$ or from $({}_3F_2)^2$.

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THE ERGODIC PROPERTY OF THE CHARACTERISTICS ON A TORUS

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1. Introduction

THE points on a torus $\mathscr{T}$ can be described by rectangular cartesian coordinates x, y both taken to modulus one. Let $X(x, y)$ and $Y(x, y)$ be real functions on $\mathscr{T}$ (so that X and Y have period unity in each variable), possessing continuous second derivatives, and not both vanishing at any point of $\mathscr{T}$. These conditions will be adhered to throughout. It then follows that the differential equations

$$\frac{dx}{dt} = X(x, y), \qquad \frac{dy}{dt} = Y(x, y) \tag{1.1}$$

define a non-singular flow on $\mathscr{T}$. Every point of $\mathscr{T}$ lies on one and only one characteristic† of (1.1). A much-studied problem, originating with Poincaré in 1885, is that of determining the long-run behaviour of the characteristics of (1.1) on $\mathscr{T}$. In this connexion there is a general conjecture by G. D. Birkhoff to the effect that the trajectories of conservative dynamical systems would be ergodic in a preponderant majority of cases: that is, the invariant measure existing for such systems would in general have the property that only sets having measure 0 or 1 transform onto themselves under the flow. It is the objective of the present note to show that the Birkhoff conjecture is true for the flows defined by a system of differential equations (1.1) without assuming *a priori* the existence of an invariant integral or measure. This is in contrast to a result due to T. Saito (6), who draws the conclusion that the flow is ergodic whenever the rotation number‡ is irrational by making use of the assumption that area is invariant. I shall, on the other hand, construct an invariant measure that is determined in a natural way by the flow (1.1) itself and prove that the flow is ergodic with respect to it.

In the interests of gaining perspective it will be useful to recall some of the earlier work that has a bearing on our problem.

† i.e. integral curve.

‡ A general reference containing relevant definitions as well as proofs of results of Denjoy and Bohl is Chapter 17 of Coddington and Levinson, *Theory of ordinary differential equations*, New York, 1955.

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In 1932 A. Denjoy proved that every characteristic of the system (1.1) is everywhere dense on the surface of the torus whenever the rotation number is irrational. In 1934 D. C. Lewis and A. Wintner, by combining a rather subtle theorem of Poincaré's with the Kronecker–Weyl equidistribution law, established a result implying that the points of intersection of any characteristic with a fixed circle $x =$ constant without contact† have an asymptotic distribution on the circle, which is the same for all characteristics, again under the hypothesis of irrationality of rotation number. This argument of course requires the existence of such a circle without contact. In 1945 C. L. Siegel showed that for the system (1.1) there always exists on $\mathscr{T}$ a simple, closed, non-bounding curve Γ with continuous curvature and without contact.

2. Outline of the argument

The idea of the following argument is quite simple. It is assumed throughout that the rotation number is irrational, which amounts to excluding periodic solutions of (1.1). In order to apply the Lewis–Wintner argument mentioned above, coordinates x, y on $\mathscr{T}$ are transformed to coordinates ξ, η on $\mathscr{T}$ so that in the (ξ, η)-coordinates (1.1) becomes

$$\frac{d\xi}{dt} = U(\xi, \eta), \qquad \frac{d\eta}{dt} = V(\xi, \eta), \tag{2.1}$$

U and V having the same properties as X and Y and, in addition, all meridian circles $\xi =$ constant being without contact. The transformed flow is exactly the same as before; only its analytical expression is altered.

In § 4 the existence of the asymptotic distribution of the points of intersection of a characteristic C with $\xi = \xi_1$ is deduced. This distribution is independent of C. In § 5 the invariant measure on $\mathscr{T}$ is generated by combining the asymptotic distributions on the various meridian circles $\xi = \xi_1$, and in § 6 the uniqueness and hence the ergodicity of this measure is established.

A number of stimulating conversations with Professor Norbert Wiener have clarified the problem at hand. In particular he constructed an argument essentially equivalent to that used in § 4.

3. A change of coordinates

By the result of C. L. Siegel's referred to above, there exists on $\mathscr{T}$ a simple closed curve Γ of class C^2 and without contact. Γ may be

† i.e., such that no characteristic is tangent to the circle $x =$ constant.

taken homotopic to the coordinate circle $x = 0$. On Γ choose a parameter η' proportional to arc length ($0 \leqslant \eta' < 1$) and let $t(\eta')$ denote the time necessary for a point $P(\eta')$ on Γ corresponding to the parameter value η' to intersect Γ again for the first time under the given motion (1.1). Let $P\{f(\eta')\}$ be this point of intersection. For each ξ on the interval $0 \leqslant \xi \leqslant 1$ a simple closed curve Γ_ξ of class C^2 is defined as follows. Γ_ξ comprises those points to which the motion (1.1) brings $P(\eta')$ in time $\xi . t(\eta')$ ($0 \leqslant \eta' < 1$): that is, Γ_ξ arises from $\Gamma \equiv \Gamma_0$ under the motion (1.1) so adjusted that all points on Γ return to Γ simultaneously. To the transform on Γ_ξ of a point $P(\eta')$ on Γ the linear coordinate

$$\eta = (1-\xi)\eta' + \xi f(\eta')$$

is assigned, instead of η', in order to obtain ξ, η as functions on $\mathscr{T}$. By known theorems on the dependence of solutions of differential equations on initial conditions it follows that $f(\eta')$ belongs to C^2.

Now under the change of coordinates on $\mathscr{T}$, $x, y \to \xi, \eta$, the equations (1.1) become

$$\frac{d\xi}{dt} = U(\xi, \eta), \qquad \frac{d\eta}{dt} = V(\xi, \eta), \tag{3.1}$$

where U and V possess continuous second derivatives and all circles $\xi =$ constant are without contact. Hence $U(\xi, \eta) \neq 0$ everywhere on $\mathscr{T}$, and it may be assumed that $U(\xi, \eta) > 0$.

4. The asymptotic distribution on $\xi = \xi_1$ ($0 \leqslant \xi_1 \leqslant 1$)

The function $f_0 = f$ introduced in § 3 defines an order-preserving homeomorphism of the circle $\xi = 0$. Let f_{ξ_1} have a similar meaning relative to the circle $\xi = \xi_1$. Let P be a point on $\xi = \xi_1$. Consider the iterates of P under f_{ξ_1},

$$P_1 = P,\ P_2 = f_{\xi_1}(P_1),\ldots,\ P_{n+1} = f_{\xi_1}(P_n),\ldots .$$

Let $\nu(N, \eta_1)$ denote the number of points among the N points $P_1, P_2,\ldots, P_N$ that lie on the arc $0 \leqslant \eta < \eta_1$ of the circle $\xi = \xi_1$.

DEFINITION. *The sequence $P_1, P_2,\ldots$ possesses an asymptotic distribution $m_{\xi_1}(\eta_1) = m(\eta_1)$ if*

$$\lim_{N\to\infty} \frac{\nu(N, \eta_1)}{N} = m(\eta_1)$$

is valid for $0 < \eta_1 \leqslant 1$, except for (at most) countably many values of η_1.

In order to be explicit I shall sketch the proof of the existence of the asymptotic-distribution function. The argument rests on the known fact† that the circular order of $P_1, P_2,\ldots,$ is exactly the same as the

† H. Poincaré, *Œuvres*, vol. 1. 149.

circular order on a circumference K of unit length of

$$\rho-[\rho],\ 2\rho-[2\rho],\ldots,\ n\rho-[n\rho],\ldots,$$

where ρ is the irrational rotation number, and $[x]$ denotes the largest integer not exceeding x. Define

$$m(P_n) = n\rho-[n\rho] \quad (n = 1, 2,\ldots). \tag{4.1}$$

Then m can be extended to a continuous monotonic mapping of $\xi = \xi_1$ onto K as follows. Let Q be any point on $\xi = \xi_1$ that is different from all P_n. Consider the set of those points among $P_1, P_2,\ldots$ lying between P_1 and Q and the complementary set of P_n's. Under the assumed conditions of regularity both sets contain infinitely many points since the set $P_1, P_2,\ldots$ is dense† on $\xi = \xi_1$. The corresponding points on K given by (4.1) define a Dedekind cut and hence a point ζ on K. Put

$$m(Q) = \zeta.$$

If Q is replaced by its coordinate η, then $m(\eta) = \zeta$, and the required mapping is defined.

Now consider the $\nu(N, \eta_1)$ points among $P_1,\ldots, P_N$ that lie on the arc $0 \leqslant \eta < \eta_1$ of the circle $\xi = \xi_1$. There are an equal number of points among $\rho-[\rho],\ldots, N\rho-[N\rho]$ lying on $0 \leqslant \zeta < \zeta_1 = m(\eta_1)$ since circular order is preserved. By the Kronecker–Weyl equidistribution theorem,

$$\lim_{N\to\infty} \frac{\nu(N, \eta_1)}{N} = \zeta_1 = m(\eta_1),$$

for all η_1.

5. The intrinsic invariant measure on $\mathscr{T}$

The asymptotic distribution m_{ξ_1} on $\xi = \xi_1$ is obviously unchanged under the time shift $t \to t+t_0$. In order to obtain a two-dimensional invariant measure on $\mathscr{T}$ consider a small arc $\Delta_1: \eta_1 \leqslant \eta < \eta_1+\delta_1$ on $\xi = \xi_1$. In time τ this arc is swept by the motion (2.1) over an element of area to which is assigned the measure

$$[m_{\xi_1}(\eta_1+\delta_1)-m_{\xi_1}(\eta_1)]\tau = m_{\xi_1}(\Delta_1)\tau, \tag{5.1}$$

and this measure is also invariant under $t \to t+t_0$ because both factors in (5.1) are invariant. In terms of (ξ, η)-coordinates the invariant measure of a Borel set R on $\mathscr{T}$ is

$$\mu(R) = \iint_R \frac{d_\eta m_\xi(\eta)\, d\xi}{U(\xi, \eta)},$$

where $d\xi$ is linear Lebesgue measure.

† Denjoy [2]. At this point we use the hypothesis that X and Y belong to C^2.

Conversely, from the two-dimensional element of measure

$$d\mu(\xi,\eta) = \frac{d_\eta m_\xi(\eta)\, d\xi}{U(\xi,\eta)},$$

the linear distributions m_ξ ($0 \leqslant \xi \leqslant 1$) can be recovered: that is,

$$m_\xi(\eta) = m_\xi(\eta) - m_\xi(0) = \frac{\int_0^\eta d_\beta m_\xi(\beta)}{\int_0^1 d_\beta m_\xi(\beta)} = \frac{\int_0^\eta U(\xi,\beta)\, d_\beta \mu(\xi,\beta)}{\int_0^1 U(\xi,\beta)\, d_\beta \mu(\xi,\beta)}.$$

It will be observed that m_ξ is related to μ as a conditional probability distribution.

Let it be recalled that in the (ξ,η)-coordinates the flow has the property that all points on any meridian circle $\xi = \xi_1$ arrive at $\xi = \xi_2$ for the first time simultaneously, say after a time interval of length τ_{12}.

Consider a (small) quasi-rectangular† set K bounded by an interval Δ_1 on $\xi = \xi_1$, the characteristics through the end points of Δ_1, and an interval Δ_2 on $\xi = \xi_2$ determined by the characteristics through the end-points of Δ_1. The *normalized* measure $\sigma(K)$ of K is

$$\sigma(K) = m_{\xi_1}(\Delta_1)\frac{\tau_{12}}{\tau_0} = m_{\xi_2}(\Delta_2)\frac{\tau_{12}}{\tau_0},$$

where τ_0 is the length of the time interval required to return to $\xi = \xi_1$ for the first time. Clearly τ_0 is independent of ξ_1.

Let χ_A be the characteristic function of a set A, and let $P = P_1, P_2, \ldots, P_N$ and $\nu(N,\Delta_1)$ have the same meaning as in § 4. Let T^t denote the flow defined on $\mathscr{T}$ by the system (1.2). Then the fraction of the total history of P that P spends in K is

$$\begin{aligned}\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \chi_K(T^tP)\,dt &= \lim_{N\to\infty}\frac{1}{N\tau_0}\int_0^{N\tau_0}\chi_K(T^tP)\,dt \\ &= \lim_{N\to\infty}\frac{1}{N\tau_0}\sum_{n=1}^{N}\tau_n\chi_{\Delta_1}(P_n) = \frac{\tau_{12}}{\tau_0}\lim_{N\to\infty}\frac{\nu(N,\Delta_1)}{N} \quad (\tau_n = \tau_{12}) \\ &= \frac{\tau_{12}}{\tau_0}m_{\xi_1}(\Delta_1) = \sigma(K). \qquad (5.2)\end{aligned}$$

6. The ergodic property of σ

It remains to establish the ergodicity of σ. It is a well-known fact that the set of all normalized invariant measures existing in a compact

† Such quasi-rectangular sets generate the class of all Borel sets by countable intersection and union.

dynamical system coincides with the convex span of the normalized ergodic invariant measures. The invariant measures are obtainable *via* F. Riesz's representation of positive linear functionals from the existence of

$$\lim_{N\to\infty}\frac{1}{N}\int_0^N \phi(T^tP)\,dt = L(\phi) = L_P(\phi)$$

for every continuous function ϕ on $\mathscr{T}$. Here P is a so-called *quasi-regular* point.† Since $L_P(\phi)$ is a positive linear functional on the continuous functions on $\mathscr{T}$, it can be written in the form

$$L_P(\phi) = \int_{\mathscr{T}} \phi(Q)\,d\mu_P(Q).$$

Here μ_P is the normalized invariant measure associated with the point P.

LEMMA. *If P is a quasi-regular point and K is a quasi-rectangular* (§ 5) *subset of $\mathscr{T}$, then*

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \chi_K(T^tP)\,dt = \mu_P(K). \tag{6.1}$$

That the limit in (6.1) exists follows (for every P) from the final paragraph of § 5. To determine its value choose an increasing sequence of continuous functions f_n and a decreasing sequence of continuous functions g_n such that

$$f_n \uparrow \chi_K \quad \text{and} \quad g_n \downarrow \chi_K \tag{6.2}$$

everywhere on $\mathscr{T}$. Then (6.2) implies

$$\lim_{n\to\infty}\int_{\mathscr{T}} (g_n - f_n)\,d\mu_P = 0, \tag{6.3}$$

and

$$\lim_{n\to\infty}\int_{\mathscr{T}} f_n\,d\mu_P = \lim_{n\to\infty}\int_{\mathscr{T}} g_n\,d\mu_P = \int_{\mathscr{T}} \chi_K\,d\mu_P = \mu_P(K). \tag{6.4}$$

On the other hand, since P is quasi-regular and f_n and g_n are continuous,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f_n(T^tP)\,dt = \int_{\mathscr{T}} f_n\,d\mu_P,$$

and similarly for g_n. Therefore

$$\int_{\mathscr{T}} f_n\,d\mu_P \leqslant \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \chi_K(T^tP)\,dt \leqslant \int_{\mathscr{T}} g_n\,d\mu_P. \tag{6.5}$$

But (6.3), (6.4), and (6.5) imply (6.1). This proves the lemma.

† J. C. Oxtoby, *Bull. American Math. Soc.* 58 (1952) 118. Oxtoby takes t discrete. There is no difficulty in developing the same theory for continuous time.

It now follows from (5.2) and the preceding lemma that $\sigma(K) = \mu_P(K)$ for every quasi-rectangular set K, and hence for every Borel set K. Here P is an arbitrary quasi-regular point: that is, there exists exactly one ergodic measure σ.

This argument proves the following

THEOREM. *If the real functions $X(x,y)$ and $Y(x,y)$ have period one in both variables, do not both vanish at any point, and possess continuous second derivatives, then there exists a unique normalized measure on the torus such that the flow defined by*

$$\frac{dx}{dt} = X(x,y), \qquad \frac{dy}{dt} = Y(x,y)$$

is ergodic with respect to this measure whenever the rotation number is irrational.

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THE ASYMPTOTIC VALUES FOR ANALYTIC FUNCTIONS WITH BOUNDED CHARACTERISTIC

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1. LET $f(z)$ denote a complex-valued function defined in the unit disk D and let P be a point on the unit circle C. We say that a is an *asymptotic value* for $f(z)$ at P if there exists a Jordan arc α contained in $D \cup C$ such that $f(z) \to a$ as $z \to P$ along α. We require $f(z)$ to be defined and continuous at all points, except P, of $\alpha \cap C$. We say that a is an *angular asymptotic value* for $f(z)$ if, in addition, α can be chosen so that it lies in some Stolz angle with vertex at P. We let $\Gamma(P,f)$ denote the set of asymptotic values and $\Gamma_\Delta(P,f)$ denote the set of angular asymptotic values for $f(z)$ at P. Obviously $\Gamma_\Delta(P,f) \subset \Gamma(P,f)$.

These sets can be defined in a natural way for a function $f(z)$ defined over an arbitrary domain in the complex plane [see (**5**) 49].

In general, the structure of the sets $\Gamma(P,f)$ is very simple for, by a remarkable theorem due to Bagemihl (**1**), given any $f(z)$ defined on D, $\Gamma(P,f)$ contains at most one value for all but an enumerable set of $P \in C$. However, at points of this exceptional set the situation may be quite complicated. For example, with a result due to Gross (**4**) and a preliminary conformal mapping, we can obtain a function $f(z)$ which is analytic in D and for which $\Gamma_\Delta(1,f)$ contains every value in the complex plane.

The following results, due to Lindelöf (**6**), describe two classes of analytic functions for which there are no exceptional sets [see also (**5**)].

THEOREM A. *Suppose that $f(z)$ is analytic and bounded in D. Then, for each P on C, $\Gamma(P,f)$ contains at most one value.*

THEOREM B. *Suppose that $f(z)$ is analytic in D and omits two finite values. Then, for each P on C, $\Gamma(P,f)$ contains at most one value.*

For a stronger form of Theorem B, due to Gross and Collingwood–Cartwright, see (**3**).

It is well known that no such conclusion is possible for the class of functions which are analytic and have bounded characteristic in D. In the present paper we shall consider just what can be said about the sets $\Gamma_\Delta(P,f)$ and $\Gamma(P,f)$ for such functions $f(z)$.

Quart. J. Math. Oxford (2), 9 (1958), 282-9.

2. Suppose that $f(z)$ is analytic in D. We say that '$f(z) \in N$' if the subharmonic function $\log^+|f(z)|$ has a finite harmonic majorant in D; N is just the class of functions which are analytic and have bounded characteristic in D. We say that '$f(z) \in N^*$' if there exists a non-negative convex function $\phi(t)$, defined for $0 \leqslant t < \infty$, such that

$$\frac{\phi(t)}{t} \to +\infty \quad \text{as} \quad t \to +\infty \tag{2.1}$$

and such that $\phi\{\log^+|f(z)|\}$ has a finite harmonic majorant in D. Clearly $N^* \subset N$. Moreover, by taking $\phi(t) = e^{pt}$, we see that, for each $p > 0$, $H_p \subset N^*$, where H_p is the class of functions which are analytic in D with

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta = O(1)$$

as $r \to 1$ (**11**).

An alternative characterization for the class N^* is suggested by the following lemma:

LEMMA 1. *Suppose that $f(z)$ is analytic in D. Then $f(z) \in N^*$ if and only if $\log^+|f(z)|$ has a harmonic majorant in D which can be written as a Poisson integral.*

Proof. Suppose that $f(z) \in N^*$ and let $w(z) = \log^+|f(z)|$. Then $w(re^{i\theta})$ has a finite radial limit $W(\theta)$ for almost all θ and, since $\phi\{w(z)\}$ has a finite harmonic majorant in D,

$$\int_{-\pi}^{\pi} \phi\{w(re^{i\theta})\} \, d\theta = O(1) \tag{2.2}$$

as $r \to 1$. Because $w(z)$ is subharmonic,

$$w(z) \leqslant \lim_{\rho \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{re}\left\{\frac{\rho e^{it}+z}{\rho e^{it}-z}\right\} w(\rho e^{it}) \, dt$$

for each $z \in D$ and hence, by (2.1), (2.2), and a theorem of de la Vallée Poussin's (**14**),

$$w(z) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\} W(t) \, dt, \tag{2.3}$$

as desired.

For the converse let $u(z)$ be the Poisson integral majorant for $w(z) = \log^+|f(z)|$ and let $U(\theta)$ be the radial limit for $u(re^{i\theta})$. We can find a non-negative, non-decreasing, convex function $\phi(t)$ satisfying (2.1) such that $\phi\{U(\theta)\}$ is Lebesgue-integrable [(**15**) 107] and from Jensen's

inequality we obtain

$$\phi\{w(z)\} \leqslant \phi\{u(z)\} \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\} \phi\{U(t)\}\, dt,$$

which completes the proof.

From the same argument we can obtain a characterization for Poisson integrals.

COROLLARY 1. *Suppose that $u(z)$ is harmonic in D. Then $u(z)$ is a Poisson integral if and only if there exists a non-negative convex function $\phi(t)$ satisfying* (2.1) *such that $\phi\{|u(z)|\}$ has a finite harmonic majorant in D.*

An immediate consequence of Corollary 1 is the following result, due to Tsuji and Ohtsuka (**10**), on the analytical invariance of Poisson integrals.

COROLLARY 2. *Suppose that $u(z)$ is a Poisson integral in D, that δ is a simply-connected domain in D, and that $z = z(\zeta)$ maps δ conformally onto D_1, the unit disk in the ζ-plane. Then $u_1(\zeta) = u\{z(\zeta)\}$ is a Poisson integral in D_1.*

3. We require the following form of an important theorem on entire functions.

LEMMA 2. *Suppose that $g(\zeta)$ is analytic in $\mu < \arg\zeta < \nu$, continuous in $\mu \leqslant \arg\zeta \leqslant \nu$, and that $\Gamma(\infty, g)$ contains n finite values. Then*

$$\liminf_{r\to\infty} r^{-\kappa} \log^+ M(r) > 0, \tag{3.1}$$

where

$$\kappa = \frac{(n-1)\pi}{\nu-\mu}, \qquad M(r) = \inf_{\mu\leqslant\theta\leqslant\nu} |g(re^{i\theta})|.$$

The proof for this version of the Denjoy–Carleman–Ahlfors theorem follows immediately from an argument due to Macintyre (**8**).

THEOREM 1. *Suppose that $f(z) \in N$. For each P on C, $\Gamma(P,f)$ contains at most two finite values. If $\Gamma_\Delta(P,f)$ contains a finite value, then $\Gamma(P,f)$ contains only one finite value.*

Proof. If $\Gamma(P,f)$ contains three finite values a, b, c there exist three Jordan arcs α, β, γ in $D \cup C$ such that $f(z) \to a$, b, c as $z \to P$ along α, β, γ, respectively. We may assume that $\alpha \cup \beta$ is a Jordan curve bounding a domain δ and that γ lies, except for P, in δ. Let $z = z(\zeta)$ map δ onto the upper half of the $\zeta = \xi + i\eta$ plane so that P goes into the point at infinity and let

$$g(\zeta) = f\{z(\zeta)\}, \qquad v(\zeta) = u\{z(\zeta)\}, \tag{3.2}$$

where $u(z)$ is the *least* harmonic majorant for $\log^+|f(z)|$ in δ. Now $u(z)$ is non-negative and harmonic in δ, and it is not difficult to see that $u(z)$ is continuous and equal to $\log^+|f(z)|$ at all points, except P, of $\alpha \cup \beta$. Hence $v(\zeta)$ is non-negative and harmonic in $\eta > 0$, continuous in $\eta \geqslant 0$, and we have

$$v(\zeta) = L\eta + \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{v(t)\,dt}{(\xi - t)^2 + \eta^2}, \tag{3.3}$$

where $L \geqslant 0$. Since $v(\zeta)$ is bounded on the ξ-axis, it follows that

$$\log^+|g(\zeta)| \leqslant v(\zeta) = O(|\zeta|), \tag{3.4}$$

uniformly as $|\zeta| \to \infty$ in $\eta \geqslant 0$. On the other hand, $\Gamma(\infty, g)$ contains a, b, c for ζ lying in $0 \leqslant \arg\zeta \leqslant \pi$, and we obtain (3.1) with $\kappa = 2$, which clearly contradicts (3.4).

For the second part of the theorem suppose that $f(z) \to a$, b as $z \to P$ along arcs α, β respectively, where a, b are finite and where β is contained in some Stolz angle with vertex at P. We can assume, without loss of generality, that $f(z)$ is continuous on C, except at P, that α is a cross-cut of D which meets C in the two points P and Q, and that β is contained, except for P, in δ, one of the two Jordan domains into which D is split by α. Let $z = z(\zeta)$ map δ onto the half plane $\eta > 0$, so that Q goes into the origin and P goes into the point at infinity. Then α goes into one half, say the positive half, of the ξ-axis and, by a well-known principle in conformal mapping [(**2**) 92], the image of β will lie in the sector $0 \leqslant \arg\zeta \leqslant \pi\rho$ for some fixed $\rho < 1$. Let $u(z)$ be the least harmonic majorant for $\log^+|f(z)|$ in δ and define $g(\zeta)$ and $v(\zeta)$ as in (3.2). As before, $v(\zeta)$ is non-negative and harmonic in $\eta > 0$ and continuous in $\eta \geqslant 0$. Thus we obtain the representation (3.3) and, since $v(\zeta)$ is bounded on the positive half of the ξ-axis, we conclude that (3.4) holds uniformly as $|\zeta| \to \infty$ in the sector $0 \leqslant \arg\zeta \leqslant \pi\rho$. But $\Gamma(\infty, g)$ contains both a and b when ζ lies in this sector, and (3.1) with $\kappa = 1/\rho > 1$ yields a contradiction.

The following result is an immediate consequence of the second part of Theorem 1.

COROLLARY 3. *Suppose that $f(z) \in N$. Then, for each P on C, $\Gamma_\Delta(P, f)$ contains at most one finite value.*

To obtain an example of a function $f(z) \in N$ with two finite values in $\Gamma(P, f)$ consider

$$g(\zeta) = \int_0^\zeta \frac{\sin t}{t}\,dt$$

in the half-plane $\eta \geqslant 0$. Here $v(\zeta) = \eta + \pi$ is a harmonic majorant for $\log^+|g(\zeta)|$ and $\Gamma(\infty, g)$ contains $\pm\frac{1}{2}\pi$ and ∞. If $\zeta = \zeta(z)$ maps D onto $\eta > 0$, so that P goes into the point at infinity, then $f(z) = g\{\zeta(z)\}$ has the desired properties.

It is a trivial consequence of the Gross–Iversen theorem (3) that, whenever $f(z)$ is analytic and $\Gamma(P,f)$ contains two finite values, then $\Gamma(P,f)$ also contains ∞.

4. If we impose additional restrictions on $f(z) \in N$, we can show that $\Gamma(P,f)$ contains at most one finite value.

THEOREM 2. *Suppose that $f(z) \in N^*$. Then, for each P on C, $\Gamma(P,f)$ contains at most one finite value.*

THEOREM 3. *Suppose that $f(z) \in N$ and that $f(z)$ omits one finite value, say 0. Then, for each P on C, $\Gamma(P,f)$ contains at most one finite value. If $\Gamma(P,f)$ contains $a \neq 0, \infty$, then $\Gamma(P,f)$ contains only a.*

In Theorem 2, the modulus of $f(z)$ is further restricted by requiring that $\phi\{\log^+|f(z)|\}$ have a harmonic majorant. This added condition is essentially a step in the direction of Theorem A, where $f(z)$ is required to be bounded. On the other hand, in Theorem 3 the additional hypothesis is on the distribution of values for $f(z)$, a step in the direction of Theorem B, where $f(z)$ is required to omit two finite values.

Proof of Theorem 2. Suppose that $\Gamma(P,f)$ contains two finite values a and b. With the aid of a preliminary conformal mapping we can assume that $f(z)$ is continuous on C, except at $P = 1$, and that

$$f(e^{i\theta}) \to a, \qquad f(e^{-i\theta}) \to b \tag{4.1}$$

as $\theta \to 0+$. By Lemma 1 and (2.3), $\log^+|f(z)|$ has a harmonic majorant in D which is the Poisson integral of $\log^+|f(e^{i\theta})|$. Hence $f(z)$ is bounded in D, and (4.1) is contradicted by Theorem A.

An alternative proof could be based on the Denjoy–Carleman–Ahlfors theorem.

Proof of Theorem 3. It is sufficient to show that, if $\Gamma(P,f)$ contains $a \neq 0, \infty$, then $\Gamma(P,f)$ contains only a. Now suppose otherwise. With the aid of a preliminary conformal mapping we can assume that $f(z)$ is continuous and non-zero on C, except at $P = 1$, and that (4.1) holds, where b may be finite or infinite. Since $f(z) \in N$ and $f(z) \neq 0, \infty$, we have

$$\log f(z) = u(z) + iv(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\gamma(t) + i\lambda, \tag{4.2}$$

where $\gamma(\theta)$ has bounded variation over $-\pi \leqslant \theta \leqslant \pi$ and λ is a real constant (**9**).

Now suppose that $\gamma(\theta)$ has a positive (negative) jump at $\theta = \theta_0$ where $-\pi < \theta_0 < \pi$. Then it is well known (**7**) that

$$u(z) \to +\infty(-\infty) \tag{4.3}$$

as $z \to e^{i\theta_0}$ radially and, since $u(e^{i\theta}) = \log|f(e^{i\theta})|$ is continuous for all $\theta \neq 0$, we conclude that $\gamma(\theta)$ is continuous except possibly at $\theta = 0$. Suppose that $\gamma(\theta)$ has a positive jump at $\theta = 0$. Then, by (4.2) and (4.3),

$$f(z) \to \infty$$

as $z \to 1$ radially, and we see that $\Gamma_\Delta(1, 1/f)$ contains 0, while $\Gamma(1, 1/f)$ contains both 0 and $1/a$. This contradicts Theorem 1. The assumption that $\gamma(\theta)$ has a negative jump at $\theta = 0$ leads to a similar contradiction, and we conclude that $\gamma(\theta)$ is continuous for $-\pi \leqslant \theta \leqslant \pi$.

From the Fatou theorem, we see that $\gamma'(\theta) = u(e^{i\theta})$ for all $\theta \neq 0$ at which $\gamma'(\theta)$ exists; hence $\gamma(\theta)$ is absolutely continuous [(**12**) 128], and $u(z)$ is a Poisson integral. It follows that we have one or both of $f(z)$ and $1/f(z)$ bounded in D, and (4.1) is contradicted by Theorem A.

5. The following example shows that the results of the previous section are in a sense best-possible.

THEOREM 4. *Suppose that $\phi(t)$ is any non-negative, non-decreasing, convex function which satisfies* (2.1). *There exists a function $f(z) \in N^*$, associated with $\phi(t)$, such that $f(z) \neq 0$ and such that $\Gamma_\Delta(1, f)$ contains both* 0 *and* ∞.

Proof. Let $U(\theta)$ be a function, continuous for $\theta \neq 0$, such that

$$U(\theta) = -U(-\theta),$$

such that $\phi\{|U(\theta)|\}$ is Lebesgue-integrable in $-\pi \leqslant \theta \leqslant \pi$, and such that

$$U(\theta) \to +\infty \tag{5.1}$$

as $\theta \to 0+$. Set $f(z) = e^{w(z)}$, where

$$w(z) = u(z) + iv(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} U(t)\, dt.$$

Now $u(re^{i\theta}) = -u(re^{-i\theta})$ for $r < 1$ and all θ. Furthermore, for $\theta \neq 0$, $u(z)$ is continuous at $z = e^{i\theta}$ with $u(e^{i\theta}) = U(\theta)$, and from (5.1) it follows that

$$u(z) \to +\infty\,(-\infty)$$

as $z \to 1$ along each ray above (below) the x-axis in D. Thus $\Gamma_\Delta(1, f)$

contains both 0 and ∞. Obviously, $f(z) \neq 0$, and with Jensen's inequality we obtain

$$\phi\{\log^+|f(z)|\} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\}\phi\{|U(t)|\}\,dt$$

which completes the proof.

The above example also suggests that in Theorems 2 and 3 no further restriction on the modulus of $f(z)$, short of asking that it be bounded, will allow us to conclude that $\Gamma(P,f)$ contains at most one value. For example, as p goes from 0 to ∞, the H_p classes form a natural link between the class N and the class of functions which are analytic and bounded in D. But, if in Theorem 4 we choose $\phi(t) = \exp(t^2)$, we obtain a function $f(z)$ such that $f(z) \in H_p$ for every $0 < p < \infty$, $f(z) \neq 0$, and $\Gamma_\Delta(1,f)$ contains both 0 and ∞.

6. All the preceding results hold for the class of functions which are meromorphic with bounded characteristic in D and which omit one value. (The classes N^* and H_p can be defined in terms of the lacunary value.) In Theorems 1 and 2, $f(z)$ will omit one value; in Theorem 3, $f(z)$ will omit two values; and the respective conclusions can then be stated in terms of these lacunary values.

Viewed in this light it is natural to inquire what can be said about the sets $\Gamma_\Delta(P,f)$ and $\Gamma(P,f)$ when $f(z)$ has no lacunary values. A partial answer to this question is given by the following result due to Valiron (**13**).

LEMMA 3. *There exists a function $g(\zeta) = h(\zeta)/j(\zeta)$ such that $h(\zeta)$ and $j(\zeta)$ are entire functions of order 1, such that*

$$\lim_{r\to\infty} g(re^{i\theta}) = G(\theta)$$

for almost all θ, and such that $G(\theta_1) \neq G(\theta_2)$ for $\theta_1 \neq \theta_2$ $(0 < \theta_1, \theta_2 < \pi)$.

Now fix $0 < \epsilon < \frac{1}{2}\pi$ and let $\zeta = \zeta(z)$ map D conformally onto Σ, the sector $\epsilon < \arg\zeta < \pi-\epsilon$, so that 1 goes into the point at infinity. With $\rho = 1+2\epsilon/\pi$ we can choose $A > 0$ so that

$$v(re^{i\theta}) = r^\rho \cos\rho(\tfrac{1}{2}\pi-\theta)+A$$

is a harmonic majorant for $\log^+|h(\zeta)|$ and $\log^+|j(\zeta)|$ in Σ. Thus $f(z) = g\{\zeta(z)\}$ is meromorphic with bounded characteristic in D and $\Gamma_\Delta(1,f)$ has the power of the continuum.

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GENERALIZED TAUBERIAN THEOREMS FOR SUMMABILITY-(A)

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1. Introduction and notation

RECENTLY Jakimovski (**4**) defined, independently of Hausdorff (**2**), the Hölder transform of order α (the (H,α)-transform) of a sequence $[s] \equiv \{s_n\}$ for any real α, as the special Hausdorff transform $\{t_n\}$ given by

$$\Delta^n t_0 = (n+1)^{-\alpha}\Delta^n s_0, \tag{1}$$

where, as usual, $\Delta^k s_n$ ($k = 1, 2, \ldots$) is the kth-order forward difference of $\{s_n\}$ beginning with s_n and $\Delta^0 s_n = s_n$. Using this definition of the (H,α)-transform, he proved the following theorems, which reduce, in the case $k = 1$, to Tauber's first theorem [(**1**) Theorem 85] and a familiar theorem of Hardy and Littlewood [(**1**) Theorem 91].

THEOREM A. *Let $\{s_n\}$ be a sequence summable-(A) to l and let there be a positive integer k such that*

$$\binom{n}{k}\Delta^k s_{n-k} = o(1) \quad (n \to \infty). \tag{2}$$

Then $\{s_n\}$ is summable $(H, -k)$ to l.†

THEOREM B. *Theorem A can be restated with $o(1)$ changed to either $O_R(1)$ or $O_L(1)$ in hypothesis* (2), *and summability $(H, -k)$ changed to summability $(H, -k+1)$ in the conclusion.*

Rajagopal (**6**) gave simple proofs of Theorems A and B, and his proof of Theorem A goes to show that the theorem is, in fact, included in one of Hausdorff's [(**2**) VI]. His proofs depend on an idea of the second author, which can be expressed as follows in a notation which will be followed throughout this note. Let $\{s_n\}$ be denoted by $[s]$ and its (H,α)-transform, i.e. $\{t_n\}$ of (1), by $H^\alpha[s]$; let $[s] = H^0[s]$ denote the

† In this note all sequences and constants are real unless they are expressly stated to be otherwise; the constants are all finite, $\{s_n\}$ is summable-(A) to l in the usual sense [(**1**) 7] and $\{s_n\}$ is summable (H, α) to l, for any real α, if the t_n of (1) tends to l.

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identity transform. Then the sequence in (2) can be written

$$\left\{\binom{n}{k}\Delta^k s_{n-k}\right\} \equiv \frac{(-1)^k}{k!}H^{-k}(H^0-H^1)(H^0-2H^1)\ldots(H^0-kH^1)[s], \quad (3)$$

where the order of the factors on the right is immaterial. Parameswaran, following this idea, was able to derive Theorem A from the simpler Theorem C which follows and of which a direct proof (as in § 4) was later kindly indicated to him by Professor H. R. Pitt in the relevant case $c > 0$.

THEOREM C. *Suppose that* $[s] = \{s_n\}$ *is summable-*(A) *to* l *and that, for some real* c, *the sequence*

$$(H^0-(c+1)H^1)[s] \text{ converges.} \quad (4)$$

Then $\{s_n\}$ *converges to* l.

For $c < 0$, Theorem C is trivial, being a well-known result of Mercer's [(1) Theorem 51] with the superfluous hypothesis of summability-(A) of $\{s_n\}$, while, for $c = 0$, Theorem C is the essential (sufficiency) part of Tauber's second theorem [(1) Theorem 86]† and, for $c > 0$, Theorem C may be deduced from an extension of Mercer's result due to Hardy [(1) Theorem 52], as in § 4.

Hardy's extension of Mercer's result leads, in fact, to the generalization of Theorem C given as Theorem C′ in § 5, and Theorem C′ in turn leads to Theorems A′ and B′ (§ 5), which are clearly generalizations of Theorems A and B in the light of (3).

One case of Theorems A′ and B′ which covers Theorems A and B, viz. the case in which the polynomial $P(t)$ of § 5 (12) has only real zeros, can be alternatively derived from Theorem I of § 3, an extension of a result due to O. Szász [(8) Lemma 5; also (12)] that a sequence is convergent if it is summable-(A) and 'quasi-monotone' according to him (10); this was originally obtained by the first author of this note for $\alpha = -1$, $c \leqslant 0$ and for $\alpha = -1$, $c = 1$.

2. Lemmas on quasi-monotone sequences

According to Szász a sequence $\{a_n\}$ is *quasi-monotone* when its terms are non-negative and there is a number c (> 0) such that

$$\left.\begin{array}{ll} \textit{either} & a_{n+1} \leqslant \left(1+\dfrac{c}{n}\right)a_n \\ \textit{or} & \left(1-\dfrac{c}{n+1}\right)a_{n+1} \leqslant a_n \end{array}\right\} \textit{ for all } n > n_0(c), \quad (5)$$

† In Tauber's case $c = 0$ of Theorem C, the convergence in (4) is necessarily to the limit 0. For, when $c = 0$, the sequence in (4) is summable-(A) to 0 by Lemma 4 of § 3.

where the second alternative, which can be shown to be equivalent to the first, is not mentioned by him. In this note we say that any real sequence is *quasi-monotone decreasing* (*q.m.d.*) *in the wide sense* if there is some real number c for which (5) holds, and *quasi-monotone increasing in the wide sense* if $\{-a_n\}$ is q.m.d. in the wide sense.† We prove in Lemma 2, which follows, the equivalence of the two alternative conditions for quasi-monotonicity and subsequently adopt the second alternative condition of (5) as the definition of an $\{a_n\}$ which is q.m.d. in the wide sense.

LEMMA 1. *If $\{a_n\}$ satisfies any one of the two alternatives of* (5), *then, for all large n, either $a_n \geqslant 0$ or $a_n \leqslant 0$.*

Proof. Irrespective of whether $\{a_n\}$ satisfies the first or the second alternative of (5), if $a_n \leqslant 0$ for some $n > \max(|c|, n_0(c))$, then $a_{n+1} \leqslant 0$ for the same n and hence also for all subsequent n.

LEMMA 2. *If $\{a_n\}$ satisfies the first alternative of* (5) *for some real* (*positive*) c, *then it satisfies the second alternative of* (5) *for some other real* (*positive*) c, *the cases of positive c in the hypothesis and the conclusion going together.*

The converse of the above statement is also true.

Proof. If $\{a_n\}$ satisfies the first alternative of (5), we have, for $n > n_1$, either $a_n \geqslant 0$ or $a_n \leqslant 0$. In the case $a_n \geqslant 0$ for $n > n_1$, given $\epsilon > 0$, we see that for

$$n > \max(n_0, n_1, |c|, \epsilon^{-1}|c(1-c-\epsilon)|)$$

we have
$$\left(1-\frac{c+\epsilon}{n+1}\right)\left(1+\frac{c}{n}\right) < 1,$$

so that
$$\left(1-\frac{c+\epsilon}{n+1}\right)a_{n+1} < \left(1+\frac{c}{n}\right)^{-1} a_{n+1} \leqslant a_n,$$

i.e. $\{a_n\}$ satisfies the second alternative of (5) with $c+\epsilon$ instead of c. In the case $a_n \leqslant 0$ for $n > n_1$, we see that, for a given $\epsilon > 0$ and for all n such that

$$n > \max(n_0, n_1, |c|, \epsilon^{-1}|c(1-c+\epsilon)|),$$

we have
$$\left(1-\frac{c-\epsilon}{n+1}\right)\left(1+\frac{c}{n}\right) > 1,$$

i.e.
$$\left(1-\frac{c-\epsilon}{n+1}\right)a_{n+1} < \left(1+\frac{c}{n}\right)a_{n+1} \leqslant a_n.$$

Thus $\{a_n\}$ satisfies the second alternative of (5) with $c-\epsilon$ instead of c.

† Our definition of quasi-monotonicity in the wide sense makes it unnecessary for us to distinguish between certain results stated by Szász separately and in pairs [e.g. (9) Lemmas 1, 2].

The proof of the second (converse) part of the lemma is similar.

LEMMA 3. *The condition for a sequence $\{s_n\}$ to be q.m.d. in the wide sense—more precisely, the second alternative of* (5) *with $\{s_n\}$ instead of $\{a_n\}$—can be written in either of the equivalent forms*

$$\left.\begin{array}{ll} \text{(i)} & (n+1)(s_{n+1}-s_n)-cs_{n+1} \leqslant 0 \quad \textit{for all large } n \\ \text{(ii)} & (H^{-1}-(c+1)H^0)[s] \leqslant 0 \quad \textit{ultimately} \end{array}\right\}. \tag{6}$$

Proof. The first alternative of (6) is obvious. To obtain the second alternative of (6), we note that the $(H,-1)$-transform of $\{s_n\}$ is the sequence $\{t_n\}$ given by

$$\frac{t_0+t_1+\ldots+t_n}{n+1} = s_n,$$

i.e.

$$t_n = (n+1)s_n - ns_{n-1}.$$

Hence $(H^{-1}-(c+1)H^0)[s]$ is the sequence

$$t_n-(c+1)s_n = (n+1)s_n-ns_{n-1}-(c+1)s_n = n(s_n-s_{n-1})-cs_n,$$

and the second alternative of (6) is seen to be a restatement of the first alternative of (6).

While Lemmas 1, 3 may be used to prove Theorem I, our Lemma 2 is needed only to present Theorem I as an extension of Szász's corresponding result for his restricted class of quasi-monotone sequences.† It may be mentioned incidentally that Szász obtains the cases $\alpha = -1$ and $\alpha = -2$ of Theorem I for his restricted class of quasi-monotone sequences in order to establish the uniform convergence of certain trigonometric series [(9) Theorems 4, 8], while he obtains the case $\alpha = 0$, $c = 0$ of Theorem I in a different context [(7) Satz II] as a useful complement to the case $\alpha = -1$, $c = 0$ (the same as the case $k = 1$ of Theorem B) which is classical.

3. Two Tauberian theorems for summability-(A)

Of our results, Theorem I includes Theorem II and its essential content can be expressed in the form of Theorem D. The principal results or ideas needed to prove the theorems are embodied in the next two lemmas, of which the first was given by Hausdorff [(2) III] and independently by Szász [(11) § 3] and the second by Rajagopal [(6) Lemma 3].

LEMMA 4. *If $[s]$ is summable-(A) to l, then so is any regular Hausdorff transform of $[s]$: in particular, so is $H^\alpha[s]$ $(\alpha > 0)$.*

† Szász's proof, however, is different from ours.

where the second alternative, which can be shown to be equivalent to the first, is not mentioned by him. In this note we say that any real sequence is *quasi-monotone decreasing* (*q.m.d.*) *in the wide sense* if there is some real number c for which (5) holds, and *quasi-monotone increasing in the wide sense* if $\{-a_n\}$ is q.m.d. in the wide sense.† We prove in Lemma 2, which follows, the equivalence of the two alternative conditions for quasi-monotonicity and subsequently adopt the second alternative condition of (5) as the definition of an $\{a_n\}$ which is q.m.d. in the wide sense.

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we have
$$\left(1-\frac{c+\epsilon}{n+1}\right)\left(1+\frac{c}{n}\right) < 1,$$

so that
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we have
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Hence $(H^{-1}-(c+1)H^0)[s]$ is the sequence

$$t_n-(c+1)s_n = (n+1)s_n-ns_{n-1}-(c+1)s_n = n(s_n-s_{n-1})-cs_n,$$

and the second alternative of (6) is seen to be a restatement of the first alternative of (6).

While Lemmas 1, 3 may be used to prove Theorem I, our Lemma 2 is needed only to present Theorem I as an extension of Szász's corresponding result for his restricted class of quasi-monotone sequences.† It may be mentioned incidentally that Szász obtains the cases $\alpha = -1$ and $\alpha = -2$ of Theorem I for his restricted class of quasi-monotone sequences in order to establish the uniform convergence of certain trigonometric series [(9) Theorems 4, 8], while he obtains the case $\alpha = 0$, $c = 0$ of Theorem I in a different context [(7) Satz II] as a useful complement to the case $\alpha = -1$, $c = 0$ (the same as the case $k = 1$ of Theorem B) which is classical.

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† Szász's proof, however, is different from ours.

LEMMA 5. *If* $[s]$ *is summable-*(A) *to* l*, and if* $H^{\alpha}[s]$ *is bounded on one side for some real* α*, then* $[s]$ *is summable* $(H, \alpha+1)$ *to* l*.*

THEOREM D. *Let* $[s]$ *be summable-*(A) *to* l *and let there be a real* c *such that the sequence*

$$(H^{-1}-(c+1)H^{0})[s] \text{ is bounded above, by } K \text{ (say)}. \tag{7}$$

Then $\{s_n\}$ *converges to* l*.*

Proof. We first show that $H^{1}[s]$ is convergent, distinguishing between the cases $c = 0$ and $c \neq 0$. In any case,

$$(H^{0}-(c+1)H^{1})[s] = H^{1}(H^{-1}-(c+1)H^{0})[s] = O_R(1) \tag{8}$$

by (7), so that, in the case $c = 0$, the sequence $(H^{0}-H^{1})[s]$ is bounded above. Also this sequence $(H^{0}-H^{1})[s]$ is summable-(A) to 0 by Lemma 4, and hence, applying Lemma 5 to it, we find that

$$H^{1}(H^{0}-H^{1})[s] = (H^{0}-H^{1})H^{1}[s]$$

converges to 0. $H^{1}[s]$ being summable-(A) by Lemma 4, we now apply Tauber's second theorem to $H^{1}[s]$ and conclude that $H^{1}[s]$ converges.

If $c \neq 0$, we write $s'_n = s_n + K/c$. Then, by Lemma 3, the statement (7) is the same as the first alternative of (6) with $\{s'_n\}$ instead of $\{s_n\}$, holding for all n, i.e. the sequence $\{s'_n\}$ is q.m.d. in the wide sense. Hence, by Lemma 1, the numbers s'_n are ultimately of the same sign; in other words, s'_n is bounded on one side and therefore so also is s_n. Lemma 5 now shows that $H^{1}[s]$ is convergent.

In any case, $H^{1}[s]$ being convergent is the same as $H^{-1}[s]$ being summable $(H, 2)$ and therefore also summable-(A). Consequently, the sequence in (7), in addition to being bounded above, is summable-(A) by Lemma 4 and hence summable $(H, 1)$ by Lemma 5, i.e. $(H^{0}-(c+1)H^{1})[s]$ is convergent, and hence $[s] = H^{0}[s]$ also is convergent, necessarily to l, as we wished to prove.

THEOREM I. *If* $[s]$ *is summable-*(A) *to* l *and there are real constants* α*,* c *such that the sequence*

$$(H^{\alpha}-(c+1)H^{\alpha+1})[s] \text{ is bounded above,} \tag{9}$$

then $[s]$ *is summable* $(H, \alpha+1)$ *to* l*.*

Proof. The case $\alpha = -1$ of Theorem I is Theorem D.

In case $\alpha > -1$, the sequence $H^{\alpha+1}[s]$ is summable-(A) to l by Lemma 4, and the result follows from the previous case with $H^{\alpha+1}[s]$ in place of $[s]$.

Finally, in case $\alpha < -1$, we have by (9) that

$$(H^{-1}-(c+1)H^{0})[s] = H^{-\alpha-1}(H^{\alpha}-(c+1)H^{\alpha+1})[s] = O_R(1).$$

Hence, by Theorem D, the sequence $[s]$ converges to l, i.e. $H^{\alpha+1}[s]$ is summable $(H, -\alpha-1)$ and therefore summable-(A) to l. The proof is now completed as in the case $\alpha > -1$.

THEOREM II. *If $[s]$ is summable-(A) to l and there are real constants α, c such that*

$$(H^{\alpha}-(c+1)H^{\alpha+1})[s] \text{ converges}, \tag{10}$$

then $[s]$ is summable (H, α) to l.

In the case $c < 0$, which is clearly reducible to Mercer's theorem, the hypothesis of summability-(A) of $[s]$ is superfluous.

Proof. The condition (10) being a particular case of (9), we first establish the convergence of $H^{\alpha+1}[s]$ using Theorem I and then that of $H^{\alpha}[s]$ using (10), the convergence of each being necessarily to the value l.

4. Alternative proofs of the preceding theorems

The alternative proofs may be presented in three stages as follows.

Direct proof of Theorem C. Excluding the known cases $c \leqslant 0$ we suppose for simplicity that the convergence in hypothesis (4) is to the limit 0. This supposition implies that

$$l = (A)\text{-}\lim s_n = 0,$$

by Lemma 4. As Professor Pitt has pointed out, our hypothesis (4), which now becomes

$$s_n-(1+c)\frac{s_0+s_1+\ldots+s_n}{n+1} = o(1),$$

implies

$$s_n = C(1+c)\frac{\Gamma(n+1)}{\Gamma(n+1-c)}+o(1) \quad (n \to \infty;\ C \text{ a constant}). \tag{11}$$

This assertion is, in fact, included in Hardy's extension of Mercer's theorem [(1) Theorem 52]. Since, in (11),

$$C(1+c)\frac{\Gamma(n+1)}{\Gamma(n+1-c)} \sim C(1+c)n^{c} \quad (n \to \infty),$$

and $[s]$ is summable-(A), we get $C = 0$, and (11) becomes the desired conclusion that $[s] = \{s_n\}$ converges to 0.

Remarks on the above proof. The proof brings out the fact that, as a result of (4) with $c > 0$, the sequence $\{s_n\}$ is the sum of a convergent sequence and a sequence asymptotically equal to $C(1+c)n^{c}$. In other words, (4) with $c > 0$ implies the convergence of $\{s_n\}$ if and only if $C = 0$, and summability-(A) is only one of many alternatives which ensure that $C = 0$; for example, we may clearly replace (A) by any totally regular method of summability.

Deduction of Theorem II from Theorem C. This is exactly like the deduction of Theorem I from Theorem D in § 3.

Deduction of Theorem I from Theorem II. Appealing to Lemma 5 with $[s]$ replaced by $(H^0-(c+1)H^1)[s]$ we find that (9) implies (10) with α replaced by $\alpha+1$, and so the conclusion of Theorem I follows from that of Theorem II.

5. Generalized Tauberian theorems for summability-(A)

In this section we generalize the theorems of §§ 1, 3 to complex sequences and complex parameters with certain exceptions.

Hardy's extension of Mercer's theorem, used in (11), establishes the validity of (11) for any complex c except for the case $\mathrm{re}\, c = 0$, $c \neq 0$. Hence we get at once the following result.

THEOREM C′. *Theorem C and Theorem II following from it are both true for a complex sequence* $[s]$, *any real* α, *and any real or complex* c *except for the case* $\mathrm{re}\, c = 0$, $c \neq 0$.

A case in which Theorem C′ is trivially true occurs when $[s]$ is a real sequence and c any complex number whatever; we tacitly ignore this case.

Theorem C′ enables us to prove our concluding two theorems in which the constants $c_1, c_2, \ldots, c_k$ (real or complex) are said to 'satisfy the condition (γ)' when the polynomial

$$P(t) = t^k + c_1 t^{k-1} + c_2 t^{k-2} + \ldots + c_k \tag{12}$$

has no zero of the form $c+1$ such that $\mathrm{re}\, c = 0$, $c \neq 0$.

THEOREM A′. *Suppose that*

(i) $[s]$ *is a complex sequence summable-*(A),

(ii) *there is a real* α *and real or complex* $c_1, c_2, \ldots, c_k$ *satisfying the condition* (γ) *and such that*

$$H^\alpha(H^0 + c_1 H^1 + c_2 H^2 + \ldots + c_k H^k)[s] \text{ converges.} \tag{13}$$

Then $[s]$ *is summable* (H, α).

Proof. Let the zeros of $P(t)$ in (12) be $a_1, a_2, \ldots, a_k$. Then (13) can be stated, after factorization, in the form

$$H^\alpha(H^0 - a_1 H^1)(H^0 - a_2 H^1)\ldots(H^0 - a_k H^1)[s] \text{ converges.}$$

The proof is by induction on k. When $k = 1$, the result reduces to Theorem C′. We assume now that the result holds for k and prove it for $k+1$. Write

$$[s'] = (H^0 - a_{k+1} H^1)[s].$$

By Lemma 4 and condition (i) we see that $[s']$ is summable-(A), and (13), in its factorized form with k changed to $k+1$, is that

$$H^{\alpha}(H^0-a_1H^1)(H^0-a_2H^1)\ldots(H^0-a_kH^1)[s] \text{ converges.}$$

Hence, by the induction hypothesis, we get that $[s']$ is summable (H,α), i.e. $H^{\alpha}(H^0-a_{k+1}H^1)[s]$ is convergent. The result now follows from Theorem C'.

The next theorem follows from the preceding, in the same manner as Theorem I from Theorem II in § 4.

THEOREM B'. *Suppose that*

(i) $[s]$ *is a real sequence summable-*(A),

(ii) *there are real* α *and real* $c_1, c_2, \ldots, c_k$ *satisfying the condition* (γ) *and such that*

$$H^{\alpha}(H^0+c_1H^1+\ldots+c_kH^k)[s] \text{ is bounded on one side.} \tag{14}$$

Then $[s]$ *is summable* $(H,\alpha+1)$.

Remarks on Theorems A', B', C'. (i) Theorem A' is a partial generalization of a result due to the first author [(**3**) Theorem 7.1].

(ii) It is an open question whether the case $\operatorname{re} c = 0$, $c \neq 0$, can be included by means of an argument different from the one adopted here or whether the case can be definitely ruled out by means of a counter-example. However, it may be pointed out that Theorems A', B', C' are true in the excluded case $c = i\beta$, $\beta \neq 0$, if we replace the hypothesis of summability-(A) by the stronger hypothesis

$$(1-z)\sum_0^{\infty} s_n z^n \text{ converges in } |z|<1 \text{ and tends to } l \text{ as } z \to 1 \text{ in } |z|<1. \tag{15}$$

It is enough to prove Theorem C' with this stronger hypothesis and with $\alpha = 0$, $c = i\beta$. The proof depends on Hardy's argument [(**1**) proof of Theorem 52], which now shows that

$$s_n - s_{n-1} = O(\log n)o(n) = o(n^2).$$

Hence by a theorem of Offord's [(**5**) Theorem 2]† we get that $[s]$ is summable $(C,2)$ or $(H,2)$, and the desired conclusion now follows from (4) with $\alpha = 0, 1$.

The authors are thankful to Professor H. R. Pitt for the direct proof of Theorem C, to Professor C. T. Rajagopal for many valuable remarks and discussions, and to the referee for showing that their arguments lead to Theorems A' and B' as well as to Theorems A and B.

† The theorem is that (15) and $s_n - s_{n-1} = o(n^k)$ $(k>0)$ together imply summability-(C,k) of $[s]$.

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A PROBLEM ON RANDOM WALK

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1. SUPPOSE a particle to start out from the origin and to move on a d-dimensional lattice taking a unit step at a time; let

$$p_{i,i+1} = \frac{1}{2d}\left(1-\frac{\epsilon}{i}\right), \qquad p_{i,i-1} = \frac{1}{2d}\left(1+\frac{\epsilon}{i}\right) \qquad (i \neq 0),$$

$$p_{i,i+1} = p_{i,i-1} \qquad (i = 0),$$

where $p_{i,i+1}$ and $p_{i,i-1}$ denote the probabilities of changing the jth coordinate ($j = 1,\ldots,d$) from i to $i+1$ and i to $i-1$ respectively. Gillis (1) has recently shown that the particle returns to the origin with probability 1 if

$$\epsilon > \tfrac{1}{2}-d^{-1},$$

whereas there is a positive probability of escape in case

$$\epsilon < \tfrac{1}{2}-d^{-1}.$$

The case $$\epsilon = \tfrac{1}{2}-d^{-1} \quad (d > 2)$$

has remained undecided, and it is the object of this note to show that the probability of return is still 1 in this limiting case $\epsilon = \frac{1}{2}-d^{-1}$.

2. The crux of the argument lies in the estimate $\gamma_n \geqslant Kn^{-1+\epsilon}$, where $0 < \epsilon < \frac{1}{2}$ and where γ_n denotes the probability of return at the $2n$th step in the one-dimensional case; this replaces the estimate

$$\gamma_n \geqslant Kn^{-1+\epsilon-\theta} \quad (\theta > 0)$$

found in (1). The remainder of the proof then proceeds precisely as in (1). K as well as the various K_i appearing in the proof denote positive constants depending on ϵ only.

It is shown in (1) that γ_n satisfies the difference equation

$$n\gamma_n = (\tfrac{1}{2}+\epsilon)\sum_{i=0}^{n-1}\gamma_i - \tfrac{1}{2}\epsilon\sum_{i=1}^{n-1}\gamma_i\gamma_{n-i} - \tfrac{1}{2}\epsilon \quad (\gamma_0 = 1), \tag{2.1}$$

from which one can in turn derive the estimate for γ_n,

$$K_1^{-1}n^{\epsilon-1-\theta} \leqslant \gamma_n \leqslant K_1 n^{\epsilon-1+\theta}, \tag{2.2}$$

where θ is an arbitrary positive number K_1 depending then on θ. Choose

$$\theta = \tfrac{1}{2}(\tfrac{1}{2}-\epsilon).$$

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Using (2.2) we get the inequalities

$$\sum_{i=1}^{n-1} \gamma_i \gamma_{n-i} \leqslant 2K_1(\tfrac{1}{2}n)^{\epsilon-\frac{1}{2}+\theta} \sum_{i=1}^{[\frac{1}{2}n]} \gamma_i \leqslant K_2 n^{\epsilon-\frac{1}{2}+\theta} \sum_{i=0}^{n-1} \gamma_i, \tag{2.3}$$

$$\sum_{i=0}^{n-1} \gamma_i \geqslant K_3 n^{\epsilon+\frac{1}{2}-\theta}. \tag{2.4}$$

Then (2.1), (2.3), and (2.4) combined yield

$$n\gamma_n \geqslant \left[(\tfrac{1}{2}+\epsilon) - \tfrac{1}{2}\epsilon K_2 n^{\epsilon-\frac{1}{2}+\theta} - \frac{\epsilon}{2K_3} n^{-\epsilon-\frac{1}{2}+\theta}\right] \sum_{i=0}^{n-1} \gamma_i$$

$$\geqslant [(\tfrac{1}{2}+\epsilon) - K_4 n^{\epsilon-\frac{1}{2}+\theta}] \sum_{i=0}^{n-1} \gamma_i. \tag{2.5}$$

We now compare the sequence $\{\gamma_n\}$ with the sequence $\{\delta_n\}$ defined as follows: $\delta_n = \gamma_n$ for $0 \leqslant n < m$, where m is the smallest positive integer for which $K_4 n^{\epsilon-\frac{1}{2}+\theta} < \frac{1}{2}(\frac{1}{2}+\epsilon)$ and

$$\delta_n = n^{-1}\{(\tfrac{1}{2}+\epsilon) - K_4 n^{\epsilon-\frac{1}{2}+\theta}\} \sum_{i=0}^{n-1} \delta_i \quad (n \geqslant m). \tag{2.6}$$

From (2.5) and (2.6) we conclude by induction that

$$\gamma_n \geqslant \delta_n \quad (n = 0, 1, \ldots)$$

and therefore proceed to estimate δ_n.

We can now rewrite (2.6) as

$$D_n = (1+a_n) D_{n-1}, \tag{2.7}$$

where $\quad a_n = n^{-1}\{(\tfrac{1}{2}+\epsilon) - K_4 n^{\epsilon-\frac{1}{2}+\theta}\}, \qquad D_n = \sum_{i=0}^{n} \delta_i.$

It follows from (2.7) that

$$\delta_{n+1} = a_{n+1} \prod_{i=m}^{n} (1+a_i) . D_{m-1} \quad (n \geqslant m). \tag{2.8}$$

Now

$$\log \prod_{i=m}^{n} (1+a_i) = \sum_{i=m}^{n} \log(1+a_i)$$

$$= (\tfrac{1}{2}+\epsilon) \sum_{i=m}^{n} \left[\frac{1}{i} + O(i^{\epsilon+\theta-\frac{1}{2}})\right] = (\tfrac{1}{2}+\epsilon)\log n + O(1),$$

since $\quad \epsilon+\theta-\tfrac{3}{2} < \epsilon + (\tfrac{1}{2}-\epsilon) - \tfrac{3}{2} < -1,$

so that $\quad \prod_{i=m}^{n} (1+a_i) \geqslant K_5 n^{\frac{1}{2}+\epsilon}. \tag{2.9}$

Using (2.8) and (2.9) we conclude that

$$\gamma_n \geqslant \delta_n \geqslant K_6 n^{-\frac{1}{2}+\epsilon}, \tag{2.10}$$

which completes the proof.

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FILTRATION OF THE HOMOTOPY GROUPS OF SPHERES

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1. Introduction

Let G be a group. By a *filtration* of G I mean a sequence of subgroups

$$G_0 \subset G_1 \subset \ldots \subset G_n \subset \ldots$$

whose intersection is the trivial subgroup and whose union is G. I also mean the function w which assigns to each element $g \in G$ the least integer n such that $g \in G_n$.

Let G be one of the homotopy groups of a sphere. The theory of reduced product spaces enables us to assign a certain filtration to G, called the *reduced product* filtration, as described in § 13 of (**4**) [see § 2 below]. An alternative filtration, which arises in connexion with the theory of Marston Morse, has been suggested by Pitcher in Problem 14 of (**8**). The suspension elements constitute G_1 in both cases, but it remains an open question as to whether the two filtrations are identical. The following observations are concerned with the reduced-product filtration exclusively. To gain an understanding of its nature we investigate the behaviour of the filtration in relation to some of the operations which are associated with homotopy groups. In § 13 of (**4**) and § 6 of (**7**), for example, the Hopf construction and the Whitehead product are discussed from this point of view. The relation with the operation of composition is particularly important, and that is our present topic. We obtain the following results.

Let $\alpha \circ \beta \in \pi_{r+1}(S^{n+1})$ denote the composition of homotopy classes $\alpha \in \pi_{m+1}(S^{n+1})$, $\beta \in \pi_{r+1}(S^{m+1})$, where $m \geqslant 1$, $n \geqslant 1$.† I shall prove that

$$w(\alpha \circ \beta) \leqslant w(\alpha).w(\beta), \tag{1.1}$$

where the full point denotes multiplication of integers. Since examples are known in which $\alpha \circ \beta = 0$ but $\alpha \neq 0$ and $\beta \neq 0$, the inequality in (1.1) cannot in general be sharpened to equality.

† The standard q-sphere is denoted by S^q, where $q = 1, 2, \ldots$. We identify S^{q+1} with the suspension of S^q in the usual way. To avoid trivialities we shall not discuss the homotopy groups of S^1.

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Let $m = 2n$, however, and suppose that α is an element of Hopf invariant unity. Then I shall prove that

$$w(\alpha \circ \beta) = 2w(\beta). \tag{1.2}$$

By Corollary 1 on p. 282 of (**11**), the elements of the form $\alpha \circ \beta$ constitute a direct summand of $\pi_{r+1}(S^{n+1})$, whose complementary summand consists of the suspension elements. Hence, if $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant unity, we can use (1.2) to express the filtration of $\pi_{r+1}(S^{n+1})$ in terms of the filtration of $\pi_{r+1}(S^{2n+1})$. For example, let $n = 1$. Then the Hopf map induces an isomorphism

$$\pi_m(S^3) \approx \pi_m(S^2) \quad (m \geqslant 3),$$

which has the effect of doubling the filtration.

Our third result was suggested by some observations of Serre's. Let $r \leqslant 4p-3$, where p is an odd prime number. By Proposition 11 on p. 285 of (**11**) there exist elements of order p in $\pi_{r+1}(S^3)$ if and only if $r = 2p-1$, $4p-4$, or $4p-3$. I shall prove the theorem:

THEOREM (1.3). *The elements of order p in $\pi_{2p}(S^3)$ and $\pi_{4p-3}(S^3)$ each have filtration $p-1$. The elements of order p in $\pi_{4p-2}(S^3)$ have filtration p.*

Thus in the case of S^3 there exist elements of as great a filtration as we please. I do not know whether the same is true of other spheres, but there are indications that it is.

In the final section I show how these and various other results enable the filtration of $\pi_{r+1}(S^{n+1})$ to be determined in many cases. I give examples including all values of r and n such that $r \leqslant n+7$.

2. The reduced-product complex

Although we are primarily concerned with the homotopy groups of spheres, the reduced product filtration is defined on a much wider class of spaces. The advantage of generalization will appear in § 5. We begin with an outline of the definition. Details will be found in § 13 of (**4**) and Part III of (**7**).

First of all, recall what is meant by a *loop-space* according to Moore (**9**). Let X be a space with a basepoint x_0. By a *loop* in X we mean a pair (f, s), where $s \geqslant 0$ and f is a map of the interval $\langle 0, s\rangle$ into X such that $f(0) = f(s) = x_0$. The composition of (f, s) with another loop (f', s') is the loop (f'', s''), where $s'' = s+s'$ and

$$f''(t) = \begin{cases} f(t) & (0 \leqslant t \leqslant s), \\ f'(t-s) & (s \leqslant t \leqslant s''). \end{cases}$$

With a suitable topology the set of loops Λ is called the *loop-space* on X. Composition of loops is continuous and determines an associative product in Λ which has x^0, the trivial loop, as unit element. Moreover, we have a natural isomorphism

$$\psi\colon \pi_r(\Lambda) \approx \pi_{r+1}(X) \quad (r \geqslant 1).$$

Next we consider reduced-product complexes as defined in (3). Let A be a countable CW-complex with precisely one 0-cell, say a^0. Then A_∞, the reduced-product complex of A, is a countable CW-complex whose points are equivalence classes of finite sequences of points in A. The sequences with m terms or fewer, where $m \geqslant 0$, determine a subcomplex denoted by A_m. We identify a^0 with A_0, which is a point-space, and we identify A with A_1. For example, let $A = S^n$. Then $A_m - A_{m-1}$ constitutes an mn-cell e^{mn}, where $m \geqslant 1$, and

$$S^n_\infty = S^n \cup e^{2n} \cup \ldots \cup e^{mn} \cup \ldots.$$

Composition of sequences determines a continuous product in A_∞ which is associative and admits a^0 as unit element.

Let I denote the interval $\langle 0, 1\rangle$. Let $\hat{A}$ denote the suspension of A, i.e. the space obtained from the product $A \times I$ by identifying

$$A \times \dot{I} \cup a^0 \times I$$

with a point. Points of $\hat{A}$ are represented by pairs (a, t), where $a \in A$ and $t \in I$, with the identification tacitly understood. Let A_Λ denote the loop-space on $\hat{A}$, based at the suspension of a^0. On A we can find a real-valued continuous function d which is positive over $A - a^0$ and zero at a^0. A non-singular map $u : A \to A_\Lambda$ is obtained from d as follows. Let $\sigma = d(a)$, where $a \in A$. If $\sigma > 0$, define $u(a)$ to be the loop (λ, σ), where

$$\lambda(\sigma t) = (a, t) \quad (t \in I).$$

If $\sigma = 0$, define $u(a)$ to be the trivial loop. Let v denote the combinatorial extension of u over A_∞, which is the map defined by

$$v(a_1 \cdot \ldots \cdot a_m) = b_1 \cdot \ldots \cdot b_m,$$

where $a_r \in A$ and $b_r = u(a_r)$ $(1 \leqslant r \leqslant m)$. Then the induced homomorphism $v_\#$, as shown below, is an isomorphism and is independent of the choice of the function d. Define ϕ to be the composition

$$\pi_r(A_\infty) \xrightarrow{v_*} \pi_r(A_\Lambda) \xrightarrow{\psi} \pi_{r+1}(\hat{A}).$$

Then ϕ is an isomorphism; we refer to it as the *canonical* isomorphism.

The reduced product filtration is defined as follows. Let $\beta \in \pi_{r+1}(\hat{A})$. Then $w(\beta) = m$, where m denotes the least integer such that $\phi^{-1}(\beta)$

contains a map with values in A_m. Notice that the filtration is an invariant of A rather than of $\hat{A}$.

3. Naturality of the filtration

We examine the behaviour of the reduced-product filtration in relation to induced homomorphisms, with the object of proving (1.1). The notation is carried over from the previous section. Suppose that we have a map

$$f : A' \longrightarrow A_\infty,$$

where A' is a countable CW-complex whose only 0-cell is mapped into a^0. Let g denote the combinatorial extension of f over A'_∞. Consider the map

$$h : \hat{A}' \longrightarrow \hat{A},$$

which is derived from f as follows. Suppose that $vf(a') = (\lambda, \sigma)$ $(a' \in A')$, where $\sigma \geqslant 0$ and λ maps $\langle 0, \sigma \rangle$ into $\hat{A}$. Then

$$h(a', t) = \lambda(\sigma t) \quad (t \in I).$$

I shall prove that

$$h_* \phi' = \phi g_*, \tag{3.1}$$

as shown in the following diagram, where g_*, h_* are homomorphisms induced by g, h, respectively, and ϕ, ϕ' are canonical isomorphisms.

$$\begin{array}{ccc} \pi_r(A'_\infty) & \xrightarrow{\phi'} & \pi_{r+1}(\hat{A}') \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \pi_r(A_\infty) & \xrightarrow[\phi]{} & \pi_{r+1}(\hat{A}) \end{array}$$

Let u', etc., mean the same in relation to A' as u, etc., mean in relation to A. Consider the map $\bar{h}$, as shown below, which is given by

$$\bar{h}(\lambda, s) = (h\lambda, s),$$

where $s \geqslant 0$ and λ maps $\langle 0, s \rangle$ into $\hat{A}'$.

$$\begin{array}{ccc} A' & \xrightarrow{u'} & A'_\Lambda \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ A_\infty & \xrightarrow[v]{} & A_\Lambda \end{array} \tag{3.2}$$

I assert that (3.2) is homotopy-commutative.† For let $\sigma' = d'(a')$, where $a' \in A'$, and let σ mean the same as in the definition of h. Let (λ_s, σ_s) denote the loop in $\hat{A}$ which is given by

$$\sigma_s = \sigma' s + \sigma(1-s), \qquad \lambda_s(\sigma_s t) = h(a', t),$$

where $s, t \in I$. The transformation of a' into (λ_s, σ_s) defines a homotopy

$$k_s : A' \to A_\Lambda$$

† Homotopies are to respect basepoints unless the contrary is stated.

such that $k_0 = vf$ and $k_1 = \bar{h}u'$. Therefore $vf \simeq \bar{h}u'$, as asserted. Let l_s denote the combinatorial extension of k_s over A'_Λ. Then l_s constitutes a homotopy of vg into $\bar{h}v'$, where v' denotes the combinatorial extension of u'.

To complete the proof of (3.1) let us consider the following diagram, where v'_*, $\bar{h}_*$ denote the homomorphisms induced by v', $\bar{h}$, respectively.

$$\begin{array}{ccccc} \pi_r(A'_\infty) & \xrightarrow{v'_*} & \pi_r(A'_\Lambda) & \xrightarrow{\psi'} & \pi_{r+1}(\hat{A}') \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \bar{h}_*} & & \downarrow{\scriptstyle h_*} \\ \pi_r(A_\infty) & \xrightarrow[v_*]{} & \pi_r(A_\Lambda) & \xrightarrow[\psi]{} & \pi_{r+1}(\hat{A}) \end{array}$$

Since $vg \simeq \bar{h}v'$, the left-hand section of the above diagram is commutative, and so is the right-hand section, by (7.3) of (7). Therefore

$$\psi v_* g_* = \psi \bar{h}_* v'_* = h_* \psi' v'_*.$$

Hence and from the definition of the canonical isomorphism we obtain (3.1).

Now suppose that $fA' \subset A_k$, where $k \geqslant 0$. We use (3.1) to prove that

$$wh_*(\beta) \leqslant k \,.\, w(\beta), \tag{3.3}$$

where $\beta \in \pi_{r+1}(\hat{A}')$ and w denotes the reduced-product filtration. Let $\eta \in \pi_r(A'_\infty)$ denote the element such that $\beta = \phi'(\eta)$. Then η contains a map

$$q : S^r \to A'_\infty$$

such that $qS^r \subset A'_j$, where $j = w(\beta)$. Since $gA'_j \subset A_{j,k}$, by the definition of g, it follows that $gqS^r \subset A_{j,k}$. However, gq represents $g_*(\eta)$, and by (3.1) we have

$$\phi g_*(\eta) = h_* \phi'(\eta) = h_*(\beta).$$

Hence (3.3) follows at once.

Finally, suppose we have an element $\alpha \in \pi_{m+1}(\hat{A})$ such that $w(\alpha) = k$. Take $A' = S^m$ in the preceding paragraph, and take f so as to represent $\phi^{-1}(\alpha)$. It follows from the definition of ϕ that α contains the map

$$h : S^{m+1} \to \hat{A}$$

which is obtained from f in the manner described. Therefore

$$h_*(\beta) = \alpha \circ \beta, \quad \text{where} \quad \beta \in \pi_{r+1}(S^{m+1}),$$

and we conclude from (3.3) that

$$w(\alpha \circ \beta) \leqslant w(\alpha) \,.\, w(\beta). \tag{3.4}$$

We obtain (1.1) from this relation by taking $A = S^n$. Special cases of (3.4) in which α or β is a suspension element can be found in § 13 of (4).

4. Proof of (1.2)

Suppose that $\pi_{2n+1}(S^{n+1})$ contains an element of Hopf invariant unity. We prove (1.2) by applying the theory of the previous section with $A = S^n$ and $A' = S^{2n}$. Let $\alpha \in \pi_{2n+1}(S^{n+1})$ have Hopf invariant unity, and let

$$f : S^{2n} \to S^n_\infty$$

be a cellular map which represents

$$\phi^{-1}(\alpha) \in \pi_{2n}(S^n_\infty).$$

Then g, the combinatorial extension of f over S^{2n}_∞, is cellular also. Consider the homomorphisms

$$\pi_r(S^n) \xrightarrow{i_*} \pi_r(S^n_\infty) \xleftarrow{g_*} \pi_r(S^{2n}_\infty),$$

where i_* is the injection and g_* is induced by g. Also consider the homomorphisms

$$\pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \xleftarrow{F} \pi_{r+1}(S^{2n+1}),$$

where E is the Freudenthal suspension and F is given by composition with α. By Corollary 1 on p. 282 of (**11**), both E and F are monomorphisms, and their images constitute a direct-sum decomposition of $\pi_{r+1}(S^{n+1})$. Also we have the relations

$$E = \phi i_*, \qquad F\phi' = \phi g_*, \tag{4.1}$$

by (10.2a) of (**4**) and (3.1) above. Therefore both i_* and g_* are monomorphisms and their images constitute a direct-sum decomposition of $\pi_r(S^n_\infty)$. Hence it follows that the map

$$k : S^n \times S^{2n}_\infty \to S^n_\infty$$

induces an isomorphism of homotopy groups, where k is given by

$$k(x, y) = x.g(y) \quad (x \in S^n,\ y \in S^{2n}_\infty).$$

Therefore k is a homotopy equivalence, by Theorem 1 of (**14**) [cf. (2.8) of (**5**)]. Choose a cellular map

$$g' : S^n_\infty \to S^{2n}_\infty$$

such that $g'g \simeq 1$. Define $F' = \phi' g'_* \phi^{-1}$, as shown in the following diagram, where g'_* is the homomorphism induced by g'.

$$\begin{array}{ccc} \pi_r(S^n_\infty) & \xrightarrow{g'_*} & \pi_r(S^{2n}_\infty) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \pi_{r+1}(S^{n+1}) & \xrightarrow[F']{} & \pi_{r+1}(S^{2n+1}). \end{array}$$

We have that $g'_* g_* = 1$ since $g'g \simeq 1$, and so

$$\phi' = \phi' g'_* g_* = F'\phi g_* = F'F\phi'.$$

by (4.1). This proves that

$$F'F = 1. \tag{4.2}$$

Let $\gamma = F(\beta)$, where $\beta \in \pi_{r+1}(S^{2n+1})$. We have to prove that

$$w(\gamma) = 2w(\beta).$$

Let j be an integer such that $w(\gamma) < 2j$. Then $\phi^{-1}(\gamma)$ can be represented by a map

$$k : S^r \to S^n_\infty$$

whose image is contained in S^n_{2j-1}. Since g' is cellular, the image of the composition

$$g'k : S^r \to S^{2n}_\infty$$

is contained in S^{2n}_{j-1}. Hence $wF'(\gamma) < j$. But $F'(\gamma) = \beta$, by (4.2). Therefore $w(\gamma) \geqslant 2w(\beta)$. Since f is cellular, we have $fS^{2n} \subset S^n_2$. Hence $w(\alpha) \leqslant 2$, and it follows from (1.1) that $w(\gamma) \leqslant 2w(\beta)$. Therefore

$$w(\gamma) = 2w(\beta),$$

which proves (1.2). Notice that (10.1) of (**5**) is practically a special case of (1.2).

5. The case of S^3

The purpose of this section is to prove (1.3). Let p be an odd prime number. By Proposition 11 on p. 285 of (**11**), neither $\pi_{2p-1}(S^{2p-3})$, $\pi_{4p-4}(S^{2p-3})$, nor $\pi_{4p-3}(S^{2p-1})$ contains an element of order p. Hence and from (4.1) of (**13**) it follows that $\pi_{2p-1}(S^2_{p-2})$, $\pi_{4p-4}(S^2_{p-2})$, and $\pi_{4p-3}(S^2_{p-1})$ are finite groups containing no elements of order p. This proves the lemma:

LEMMA (5.1). *The elements of order p in $\pi_{2p}(S^3)$ and $\pi_{4p-3}(S^3)$ are of filtration at least $p-1$. The elements of order p in $\pi_{4p-2}(S^3)$ are of filtration at least p.*

Hence and from (9.3) of (**5**) it follows that the elements of order p in $\pi_{2p}(S^3)$ have filtration $p-1$ exactly. Moreover, the composition of such an element with its $(2p-3)$-fold suspension is of filtration at most $p-1$, by (1.1). Therefore it follows from (5.1) and the second remark below the Corollary on p. 285 of (**11**) that the elements of order p in $\pi_{4p-3}(S^3)$ have filtration $p-1$ exactly. This proves the first part of (1.3).

To prove the second part, let $\alpha \in \pi_{2p-1}(S^2_\infty)$ and $\gamma \in \pi_{4p-3}(S^2_\infty)$ denote elements of order p. Construct a complex A' by attaching a $2p$-cell to

S^{2p-1} by a map of degree p. Then a cellular map of S^{2p-1} into S^2_∞ which represents γ can be extended to a cellular map

$$f : A' \to S^2_\infty.$$

Consider the map $$h : \hat{A}' \to S^3,$$

which is obtained from f as in § 3 with $A = S^2$. By Proposition 8 on p. 283 of (**7**), the induced homomorphism

$$h_* : \pi_{r+1}(\hat{A}') \to \pi_{r+1}(S^3)$$

maps onto the p-primary component if $r \leqslant 4p-2$. Therefore $\pi_{4p-2}(\hat{A}')$ contains an element β such that $h_*(\beta) = \phi(\gamma)$. Hence

$$w\phi(\gamma) \leqslant p.w(\beta),$$

by (3.3). But $$\beta \in E\pi_{4p-3}(A'),$$

by the Blakers–Massey suspension theorem [see (2.6) of (**12**)]. Hence $w(\beta) = 1$, and so $w\phi(\gamma) \leqslant p$. Therefore the elements of order p in $\pi_{4p-2}(S^3)$ are of filtration at most p. Taken in conjunction with (5.1) this completes the proof of (1.3).

6. Examples

I do not propose to discuss here the problem of deciding whether a given element $\alpha \in \pi_{r-1}(S^{n+1})$ is a suspension element or not. Wherever computations of homotopy groups are available, this information is included. Our interest lies in cases where $w(\alpha) > 2$. For, when $w(\alpha) \leqslant 2$, we have that $w(\alpha) = 2$ if and only if α is not a suspension. Thus we concentrate attention on the range $r > 3n$, because

$$w(\alpha) \leqslant 2 \quad \text{if} \quad r \leqslant 3n,\dagger \tag{6.1}$$

by the cellular-approximation theorem. I postpone the cases $n = 1$, 3, 7 since (1.2) applies to these. Let us compute the filtration when $r \leqslant n+7$. In view of the above remarks, the only groups which need to be examined are $\pi_{r+1}(S^3)$, where $r = 7, 8, 9$.

First, let $\alpha \in \pi_8(S^3)$. Then $\alpha = \beta \circ \gamma$, where $\beta \in \pi_6(S^3)$ and $\gamma \in \pi_8(S^6)$, by (3.8) of (**1**) and (*f*) on p. 81 of (**12**). We have $w(\beta) \leqslant 2$, by (6.1), and $w(\gamma) \leqslant 1$ since γ is a suspension. Hence $w(\alpha) \leqslant 2$, by (1.1).

Secondly, let $\alpha \in \pi_9(S^3)$. Then $3\alpha = 0$, as in (**1**), and it follows at once from (1.3) that $w(\alpha) \leqslant 2$.

Thirdly, consider $\pi_{10}(S^3)$, which is cyclic of order 15 according to (**10**).‡ Let γ denote one of its generators. It follows from (1.3) that

† The range in which $w(\alpha) \leqslant 2$ can be somewhat extended, especially when n is odd.

‡ Details not given in (**10**) can be supplied from (**5**).

$w(m\gamma) = 4$ if m is prime to 5, and that $w(m\gamma) = 3$ for all other values of m which are not multiples of 15.

If $r \leqslant 8$, therefore, $\pi_{r+1}(S^3)$ contains no element of filtration greater than 2. It is easy to show by means of the published calculations that elements of filtration 2 occur when $r = 5,\ldots, 8$. Hence it follows from (1.2) that $\pi_{r+1}(S^2)$ contains elements of filtration 4 for the same values of r. Suppose that $w(\alpha) \geqslant 3$, where $\alpha \in \pi_{r+1}(S^{n+1})$ and $r \leqslant n+7$. Then it follows from (6.1) and the above results that either $n = 1$ and $5 \leqslant r \leqslant 8$, or $n = 2$ and $r = 9$. I leave the details to be filled in by the reader. Difficulties are encountered in the case of $\pi_{11}(S^3)$, and it appears that further analysis will be required before we can deal with the 8-stem.

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THEOREMS ON SOME METHODS OF SUMMABILITY

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1. Introduction

I COMMENCE with a description of notation and terminology and give definitions of some of the matrices and methods of summability considered in this note.

Suppose throughout that s, s_n $(n = 0, 1, \ldots)$ are arbitrary complex numbers, that $\lambda \geqslant -1$, and that

$$\epsilon_n^\alpha = \binom{n+\alpha}{n}.$$

The method A_λ. Write $s_n \to s\ (A_\lambda)$ if either

(i) $\lambda > -1$ and $(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n \to s$ as $x \to 1-$

or (ii) $\lambda = -1$ and $-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} \to s$ as $x \to 1-$.

The Hausdorff matrix H. Let $\{\mu_n\}$ be a sequence of real numbers and let

$$t_n = \sum_{r=0}^{n} s_r \binom{n}{r} \sum_{\nu=0}^{n-r} (-1)^\nu \binom{n-r}{\nu} \mu_{r+\nu}.$$

Denote the matrix of the linear transformation from $\{s_n\}$ to $\{t_n\}$ by H, and write $H(s_n)$ for t_n. Then H is a Hausdorff matrix which is said to be 'generated by the sequence $\{\mu_n\}$'. I use the same symbol for both matrix and associated Hausdorff summability method, i.e. I write $s_n \to s\ (H)$ to mean that $H(s_n) \to s$. If $\mu_n \neq 0$, the Hausdorff matrix generated by $\{1/\mu_n\}$ is denoted by H^{-1}.

The product method $A_\lambda H$. Write $s_n \to s\ (A_\lambda H)$ if $H(s_n) \to s\ (A_\lambda)$.

The Hölder matrix (H, α). For any real α, this is the Hausdorff matrix generated by the sequence $\{(n+1)^{-\alpha}\}$.

The Cesàro matrix (C, α) $(\alpha > -1)$. This is the Hausdorff matrix generated by $\{1/\epsilon_n^\alpha\}$. In the specified range only, it is the matrix of the Cesàro method (C, α) [Hardy (**6**) 251]; and

$$(C, \alpha)(s_n) = \frac{1}{\epsilon_n^\alpha} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r.$$

The matrix (C^*, α). I define this to be the Hausdorff matrix generated by $\{1/\epsilon_n^{\alpha}\}$ when $\alpha > -1$, and by $\{\epsilon_n^{-\alpha}\}$ when $\alpha \leqslant -1$, so that

$$(C^*, \alpha) = \begin{cases} (C, \alpha) & (\alpha > -1), \\ (C, -\alpha)^{-1} & (\alpha \leqslant -1). \end{cases}$$

The matrix (C, α, β) $(\beta > -1,\ \alpha+\beta > -1)$. I use the notation

$$s_n^{\alpha,\beta} = \frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^{\beta} s_r,$$

and denote by (C, α, β) the matrix of the linear transformation from $\{s_n\}$ to $\{s_n^{\alpha,\beta}\}$. Note that, for $\alpha > -1$, $(C, \alpha, 0) = (C, \alpha)$.

Suppose that P and Q are summability methods (or Hausdorff matrices). P is said to be *regular* if $s_n \to s$ (P) whenever $s_n \to s$. If $s_n \to s$ (P) whenever $s_n \to s$ (Q), P is said to *include* Q, and we write $P \supseteq Q$. If $P \supseteq Q$ and $Q \supseteq P$, P and Q are said to be *equivalent*, and we write $P \simeq Q$.

The following results are known.

(I) *If H is a regular Hausdorff matrix, then $A_\lambda H \supseteq A_\lambda$.*

[$\lambda = 0$, Hausdorff (**7**) 191; see also Szász (**11**); $\lambda > -1$, Amir (**2**) 376; see also Borwein (**3**) 222; $\lambda = -1$, Borwein (**5**) 218.†]

(II) $A_\alpha \supseteq A_\beta \supseteq (C, \gamma)$ $(\beta > \alpha \geqslant -1;\ \gamma > -1)$.

[Borwein (**3**) and (**4**) 348.]

(III) $A_\lambda(C, \beta) \supseteq A_\lambda(C, \alpha) \supseteq (C, \gamma)$ $(\lambda = 0;\ \beta > \alpha > -1;\ \gamma > -1)$.

[Amir (**1**); see also Lord (**8**) 243.]

In § 2 of this note the result (I) is extended; also, conditions are obtained which are sufficient for $A_\lambda H$ to include K when H and K are Hausdorff methods.

In § 3 it is proved that $(C^*, \alpha) \simeq (H, \alpha)$ for all real α; and that, for $\beta > -1$, $\alpha+\beta > -1$, (C, α, β) is a Hausdorff matrix equivalent to (C^*, α).

In § 4 the results of § 2 and § 3 are applied to the method $A_\lambda(C^*, \alpha)$. It is proved, *inter alia*, that (III), with C^* in place of C, is true whenever $\lambda \geqslant -1$, $\beta > \alpha$, and that the methods $A_{-\alpha}$, $A_0(C^*, \alpha)$ are equivalent for all $\alpha < 1$.

2. The method $A_\lambda H$

Suppose in what follows that H, K are Hausdorff matrices generated respectively by the real sequences $\{\mu_n\}$, $\{\nu_n\}$, and denote by HK both the matrix product and the associated summability method. It is

† See the concluding remarks.

familiar that HK is a Hausdorff matrix generated by the sequence $\{\mu_n \nu_n\}$ and, consequently, that $HK = KH$. Further, it is known and easily proved that, when $\mu_n \neq 0$, $K \supseteq H$ if and only if KH^{-1} is regular.

The above results [for proofs see Hardy (**6**) ch. xi] are used without special mention in the rest of the paper.

THEOREM 1. *If $\mu_n \neq 0$ and $A_\lambda \supseteq KH^{-1}$, then $A_\lambda H \supseteq K$.*

Proof. Suppose that $s_n \to s\ (K)$ and let $t_n = H(s_n)$. Then

$$K(s_n) = KH^{-1}(t_n) \to s, \quad \text{so that} \quad t_n \to s\ (KH^{-1}).$$

It follows that $t_n \to s\ (A_\lambda)$, i.e. $s_n \to s\ (A_\lambda H)$; and the proof is complete.

THEOREM 2. *If either* (i) *H is regular or* (ii) *$\mu_n \neq 0$ and $A_\lambda \supseteq H^{-1}$, then $A_\lambda H$ is regular.*

Proof. Since, by (II), A_λ is regular, it follows from (I) that $A_\lambda H$ is regular when (i) is satisfied. When (ii) is satisfied the result follows from Theorem 1.

The next theorem is an extension of (I).

THEOREM 3. *If H is regular, then $A_\lambda KH \supseteq A_\lambda K$.*

Proof. Suppose that $s_n \to s\ (A_\lambda K)$, so that $t_n = K(s_n) \to s\ (A_\lambda)$. Hence, by (I), $H(t_n) = KH(s_n) \to s\ (A_\lambda)$. The theorem follows.

An immediate corollary of Theorem 3 is the theorem:

THEOREM 4. *If $K \supseteq H$ and $\mu_n \neq 0$, then $A_\lambda K \supseteq A_\lambda H$.*

3. The matrices (H, α), (C^*, α), (C, α, β)

It is known that the Hölder matrix (H, δ) is regular for $\delta \geqslant 0$ [(**6**) ch. xi], and it is evident that, for all real α, β,

$$(H, \alpha)(H, \beta) = (H, \alpha+\beta).$$

Hence, for $\delta > 0$ and all real α,

$$(H, \alpha+\delta) \supseteq (H, \alpha).$$

Further, it is known [(**6**) Theorem 211] that, for $\alpha > -1$,

$$(H, \alpha) \simeq (C, \alpha).$$

Since, for $\alpha > -1$, $(C^*, \alpha) = (C, \alpha)$; and, for $\alpha \leqslant -1$,

$$(C^*, \alpha) = (C, -\alpha)^{-1} \simeq (H, -\alpha)^{-1} = (H, \alpha),$$

we obtain

THEOREM 5. *For all real α, $(C^*, \alpha) \simeq (H, \alpha)$.*

Immediate consequences are the theorems:

THEOREM 6. *For $\alpha > \beta$, $(C^*, \alpha) \supseteq (C^*, \beta)$.*

THEOREM 7. *For all real α, β, $(C^*, \alpha)(C^*, \beta) \simeq (C^*, \alpha+\beta)$.*

The cases $\alpha > -1$ of Theorem 5 and $\alpha > \beta > -1$ of Theorem 6 are, of course, standard. The cases $\alpha = -1, -2, \ldots$ of Theorem 5, $-1 \geqslant \alpha > \beta$ of Theorem 6, and $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$ of Theorem 7 have been given respectively by Lyra (**9**), Mears (**10**), and Zygmund (**12**).

We now prove

THEOREM 8. *If* $\beta > -1$, $\alpha+\beta > -1$, *then*

$$(C, \alpha, \beta) = (C, \alpha+\beta)(C, \beta)^{-1},$$

so that (C, α, β) *is the Hausdorff matrix generated by the sequence* $\{\epsilon_n^\beta / \epsilon_n^{\alpha+\beta}\}$.

Proof. It is well known and easily verified that, for $\beta > -1$, $\alpha+\beta > -1$,

$$\frac{1}{\epsilon_n^{\alpha+\beta}} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^\beta s_r^{\beta,0} = s_n^{\alpha+\beta,0},$$

whence $(C, \alpha, \beta)(C, \beta) = (C, \alpha+\beta)$. The theorem follows.

In consequence of Theorems 5 and 8 we have

THEOREM 9. *For* $\beta > -1$, $\alpha+\beta > -1$, $(C, \alpha, \beta) \simeq (C^*, \alpha)$.

The case $\alpha > -1$, $\beta > -1$, $\alpha+\beta > -1$ of this result was proved in essence by Zygmund (**12**).

So far we have considered the Cesàro method (C, α) only in the range $\alpha > -1$. The standard definition† of this method in the range $\alpha \leqslant -1$ is:

$$s_n \to s\ (C, \alpha) \quad \text{if} \quad \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r = s\epsilon_n^\alpha + o(n^\alpha) \quad \text{and} \quad s_n \to s\ (A_0).$$

This definition is due to Hausdorff (**7**) who proved the methods (C, α) and (H, α) to be equivalent (for all real α).

Hence, in virtue of Theorem 5, we have

THEOREM 10. *The methods* (C, α) *and* (C^*, α) *are equivalent for all real* α.

The case $\alpha = -1, -2, \ldots$ of this result has been proved by Lyra [(**9**), Satz 1].

4. The method $A_\lambda(C^*, \alpha)$

In what follows I use the notations

$$s_n^\alpha = s_n^{*\alpha} = \frac{1}{\epsilon_n^\alpha} \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} s_r \quad (\alpha > -1),$$

$$s_n^{*\alpha} = \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \epsilon_r^{-\alpha} s_r \quad (\alpha \leqslant -1).$$

I also use the first notation with a in place of s.

† For a full discussion of this and other definitions see Lyra (**9**).

Note that $s_n^\alpha = (C,\alpha)(s_n)$ $(\alpha > -1)$, and, in view of Theorem 8 with $\beta = -\alpha$, $s_n^{*\alpha} = (C^*,\alpha)(s_n)$ (all real α). [Cf. Mears (**10**), (11).]

The next theorem generalizes (III).

THEOREM 11. *If $\alpha > \beta$ and γ is real, then*

$$A_\lambda(C^*,\alpha) \supseteq A_\lambda(C^*,\beta) \supseteq (C^*,\gamma).$$

Proof. The first inclusion is a consequence of Theorems 4 and 6. Further, in virtue of (II) and Theorems 6 and 7, we have

$$A_\lambda \supseteq (C^*,\gamma-\beta) \simeq (C^*,\gamma)(C^*,\beta)^{-1},$$

so that, by Theorem 1, $A_\lambda(C^*,\beta) \supseteq (C^*,\gamma)$.

A corollary of Theorem 11 is that $A_\lambda(C^*,\alpha)$ is regular for all real α. The next theorem shows that for $\beta > -1$ the strength of the method $A_\beta(C^*,\alpha+\beta)$ is independent of β.

THEOREM 12. *If $\beta > -1$ and α is any real number, then*

$$A_0(C^*,\alpha) \simeq A_\beta(C^*,\alpha+\beta).$$

Proof. It follows from the formal identity

$$\sum_{n=0}^{\infty} s_n x^n = (1-x)^\beta \sum_{n=0}^{\infty} \epsilon_n^\beta s_n^\beta x^n \quad (0 < x < 1;\ \beta > -1),$$

that, if one of the series is convergent throughout the interval $(0,1)$, then so is the other. Consequently, $s_n \to s$ (A_0) if and only if $s_n^\beta \to s$ (A_β); and so $s_n^{*\alpha} \to s$ (A_0) if and only if $(C^*,\beta)(s_n^{*\alpha}) \to s$ (A_β). Hence,

$$A_0(C^*,\alpha) \simeq A_\beta(C^*,\alpha)(C^*,\beta)$$

and, since $(C^*,\alpha)(C^*,\beta) \simeq (C^*,\alpha+\beta)$, application of Theorem 4 yields the required result.

Put $\beta = -\alpha$ in the above theorem to get the corollary

$$A_0(C^*,\alpha) \simeq A_{-\alpha} \quad (\alpha < 1),$$

i.e. for $\alpha < 1$,

$$(1-x)\sum_{n=0}^{\infty} s_n^{*\alpha} x^n \to s \quad \text{as} \quad x \to 1-$$

if and only if

$$(1-x)^{1-\alpha}\sum_{n=0}^{\infty} \epsilon_n^{-\alpha} s_n x^n \to s \quad \text{as} \quad x \to 1-.$$

The final two theorems are concerned with the method $A_{-1}(C,\alpha)$.

Suppose that $\qquad s_n = \sum_{r=0}^{n} a_r \quad (n = 0,1,\ldots).$

For $\alpha > 0$, I define the logarithmic method of summability (L,α) as follows:

$$s_n \to s\ (L,\alpha) \quad \text{or} \quad \sum_{n=0}^{\infty} a_n = s\ (L,\alpha)$$

if $$-\frac{1}{\log(1-x)}\sum_{n=0}^{\infty} a_n^{\alpha} x^{n+1} \to \alpha s \quad \text{as} \quad x \to 1-.$$

Note that the methods $(L, 1)$ and A_{-1} are identical.

We require two lemmas, of which the first is known [(5) Theorem 2] and the second is a simple consequence of a known result [(5) Lemma 1].

LEMMA 1. $A_{-1} \supseteq A_0(C, 1)$.

LEMMA 2. *If* $p > 0$, $q > 0$ *and* $s_n \to s\ (A_{-1})$, *then*

$$\frac{n+p}{n+q} s_n \to s\ (A_{-1}).$$

THEOREM 13. *For* $\alpha > 0$, $(L, \alpha) \simeq A_{-1}(C, \alpha-1)$.

Proof. Since $s_n \to s\ (L, \alpha)$ if and only if $(n+1)a_n^{\alpha} \to \alpha s\ (A_{-1})$, the required result is a consequence of Lemma 2 and the easily verified identity

$$(n+1)a_n^{\alpha} = \frac{n+1}{n+\alpha}\alpha s_n^{\alpha-1} \quad (\alpha > 0).$$

THEOREM 14. *For* $\alpha > \beta > 0$, $(L, \alpha) \supseteq (L, \beta) \supseteq A_0(C, \beta)$.

Proof. In virtue of Theorems 11 and 13,

$$(L, \alpha) \supseteq (L, \beta) \simeq A_{-1}(C, \beta-1).$$

Further, by Lemma 1 and Theorems 4 and 7,

$$A_{-1}(C, \beta-1) \supseteq A_0(C, 1)(C, \beta-1) \simeq A_0(C, \beta).$$

This completes the proof.

I am indebted to the referee for suggestions which enabled me to simplify the presentation of the material in § 3, and also for supplying most of the references there given.

Remarks on results (I) (added 10 July 1958). Professor C. T. Rajagopal has kindly sent me a reprint of his paper 'Product of two summability methods', *J. Indian Math. Soc.* 18 (1954) 89–105. This paper, which I had not seen previously, predates my paper (5) but not Amir's paper (2). In it Professor Rajagopal deduces, from his Theorem 1, the case $\lambda > -1$ of (I), and also (without giving details) a result equivalent to the following:

if H *is a regular Hausdorff matrix and* $t_n = H(s_n)$, *then* $t_{n+1} \to s\ (A_{-1})$ *whenever* $s_{n+1} \to s\ (A_{-1})$.

This is proved directly in (5) and is the key result used there in establishing the case $\lambda = -1$ of (I).

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HOMOLOGY WITH ZERO COEFFICIENTS

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Introduction

By a *homology theory* we mean one which satisfies the axioms 1–7 in Chapter I of (2), and is defined on the category of topological pairs. We construct examples of non-trivial homology theories whose coefficient group is zero. Some of these theories may appear to be rather artificial, but one of them arises naturally out of the study of compression problems.

1. The first example

Let H' denote the singular homology theory with integral coefficients. Let $T(X, A)$ denote the strong direct sum of the groups $H'_m(X, A)$, where (X, A) is a topological pair, and let $S(X, A)$ denote the weak direct sum. We regard $S(X, A)$ as a subgroup of $T(X, A)$, in the usual way, and denote the corresponding factor group by $U(X, A)$. Define a homology theory, H, as follows. Write $H_n(X, A) = U(X, A)$, for all values of n. Let

$$\partial: H_n(X, A) \to H_{n-1}(A)$$

be determined by the direct sum of the boundary homomorphisms of H', and let induced homomorphisms be similarly defined. Verification of the axioms is straightforward. Notice that $U(X, A) = 0$ if and only if $H'_m(X, A) = 0$ for all sufficiently large m. Hence H is zero on pairs of finite-dimensional polyhedra, and in particular has zero coefficient group. But H is not zero on the infinite complex projective space, for example.

2. On co-cochains

In defining the obstruction homology class for the compression problem, as in (3), one is led to consider 'chain-groups' of the form

$$\bar{C}_n(G) = \mathrm{Hom}\{\mathrm{Hom}(C_n, Z), G\},$$

where C_n is a free abelian chain group, G is an (abelian) group of coefficients, and Z denotes the group of integers. A 'chain-mapping'

$$\xi: C_n \otimes G \to \bar{C}_n(G)$$

is defined by $(\xi(c \otimes g))\theta = \theta(c)g$, where $\theta: C_n \to Z$. If C_n or G is finitely-generated, then ξ is an isomorphism [see (4)]. But, if, for example,

$G = R$, the group of rationals, and if the rank of C_n is infinite, then $\xi(C_n \otimes R) \neq \bar{C}_n(R)$. We have an exact sequence

$$0 \to C_n \otimes R \xrightarrow{\xi} \bar{C}_n(R) \xrightarrow{\eta} C''_n(R) \to 0,$$

where $C''_n(R)$ is defined as the cokernel of ξ, and η denotes the natural projection. Let C_n be the group of n-chains in the singular theory. Then we have an exact sequence

$$\ldots \to H'_n(X, A) \to \bar{H}_n(X, A) \to H''_n(X, A) \to H'_{n-1}(X, A) \to \ldots,$$

where H', $\bar{H}$, H'' are the homology theories defined by means of $C_n \otimes R$, $\bar{C}_n(R)$, $C''_n(R)$, respectively. Thus, H' is the singular theory with rational coefficients, $\bar{H}$ is a theory based on 'co-cochains', and H'' is a non-trivial homology theory with zero coefficient group.

3. A process of construction

Let $\mathscr{A}$ be a category and let $\mathscr{A}'$ be an exact category, in the sense of (1). Suppose that we have natural transformations

$$S \xrightarrow{\xi} T \xrightarrow{\eta} U,$$

where S, T, U are (covariant) functions from $\mathscr{A}$ to $\mathscr{A}'$. We describe (ξ, η) as an *exact* pair of natural transformations if and only if

$$0 \to SA \xrightarrow{\xi_A} TA \xrightarrow{\eta_A} UA \to 0$$

is exact for all $A \in \mathscr{A}$. Suppose that $\mathscr{A}$ is also an exact category. Recall that an exact function is one which transforms exact sequences into exact sequences. By (5.5) of (1) and the dual thereof we have

LEMMA (3.1). *Let T be an exact functor and let (ξ, η) be an exact pair of natural transformations. Then S is exact if and only if U is exact.*

We say that ξ is *monomorphic* if ξ_A is always a monomorphism. We say that η is *epimorphic* if η_A is always an epimorphism. Notice that ξ is monomorphic and η is epimorphic if (ξ, η) is an exact pair.

Let $\mathscr{G}$ denote the category of (additive) abelian groups, which is an exact category. Let S, $T: \mathscr{G} \to \mathscr{G}$ be covariant functors, and let $\xi: S \to T$ be a natural transformation. Let $U: \mathscr{G} \to \mathscr{G}$ denote the functor such that UA is the cokernel of ξ_A, where $A \in \mathscr{G}$, and such that the usual projection $\eta: T \to U$ is a natural transformation. We refer to U as the 'functor determined by ξ'. We have at once

THEOREM (3.2). *Suppose that ξ is monomorphic. Then (ξ, η) is an exact pair of natural transformations. Hence U is exact if S and T are exact.*

A similar result is obtained when we take an epimorphic natural transformation $\eta\colon T \to U$, and use it to define S.

Let $T\colon \mathscr{G} \to \mathscr{G}$ be a covariant functor. We have

THEOREM (3.3). *Let H be a homology theory and suppose that T is exact. Then the images under T of the groups and homomorphisms of H constitute a homology theory TH.*

The proof that TH satisfies the axioms is straightforward and will be omitted. These two theorems enable us to construct further examples of non-trivial homology theories with zero coefficient group, as shown in the following section.

4. A further example

Let R denote the group of rationals, and let $S, K\colon \mathscr{G} \to \mathscr{G}$ denote the usual functors such that $SA = A \otimes R$, $KA = \mathrm{Hom}(A, R)$. Define $T = KKS$, so that

$$TA = \mathrm{Hom}\{\mathrm{Hom}(A \otimes R, R), R\}.$$

Then T, like S, is covariant, and a natural transformation $\xi\colon S \to T$ is given by

$$(\xi_A x)\theta = \theta x,$$

where $x \in A \otimes R$ and $\theta\colon A \otimes R \to R$. Since R is without torsion and is infinitely divisible, the functors S and K are exact. Hence T is exact. Also ξ is monomorphic, since

$$\mathrm{Hom}(A \otimes R, R) = \mathrm{Hom}_R(A \otimes R, R),$$

where $A \otimes R$ is regarded as a vector-space over R. Hence U is exact, by (3.2), where $U\colon \mathscr{G} \to \mathscr{G}$ is the covariant functor determined by ξ. Also $UZ = 0$, where Z is the group of integers, since $\xi_Z(SZ) = TZ$.

Let H' denote the singular homology theory with integral coefficients. Then UH' is a homology theory, by (3.3), and has zero coefficient group since $UZ = 0$. Let Y be an infinite discrete space of power α, say. The power of $H'_0(Y) \otimes R$ is also α, while the power of $TH'_0(Y)$ is 2^{2^α}. Hence $UH'_0(Y) \neq 0$.

Since $UH'_m(P) = 0$ for all m when P is a single point, we obtain a homology theory H by defining

$$H_n(X, A) = \sum_{-\infty}^{\infty} UH'_{n+i}(X, A),$$

for all n, where $\sum$ denotes weak direct summation over i, with corresponding definitions for the functorial homomorphisms. The coefficient group is zero since $H_0(P) = 0$. But

$$H_n(Y) = UH'_0(Y) \neq 0,$$

where Y is an infinite discrete space.

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